Discrete Structures	Homework 2
CS2800 Fall 2014	More probability

1. Alice, Bob and Carol share an apartment. One day, their fire alarm goes off and the fire brigade arrives to find the apartment empty and a pan left smoking on the lit stove. A neighbor tells the fire marshal that Bob is likely to be the culprit, since he is the most absent-minded and forgets to turn off the stove half the time he uses it. Alice forgets to turn it off only a third of the time she uses it, and Carol only a quarter of the time.

   (a) Not knowing how often each of the three uses the kitchen to cook, the fire marshal assigns a uniform prior of 1/3 to the possibility that each was cooking. In the fire marshal's opinion, what is the probability Bob left the stove on?

   (b) Dave, who is a close friend of the trio, knows that Alice does most of the cooking in the apartment, Carol hardly ever cooks (1% of the time), and Bob cooks occasionally (15% of the time). According to Dave, what is the probability Bob left the stove on?

   Show the reasoning and calculations for your answers.

2. If $B \subseteq A$ and $C \subseteq A$, and none of $P(A)$, $P(B)$ and $P(C)$ is zero, then show that

$$\frac{P(C)}{P(B)} = \frac{P(C \mid A)}{P(B \mid A)}$$

3. We often want to perform two experiments and wish to reason about the combined outcomes of the two. For example, the first experiment could be a coin toss, and the second a dice roll, and we might ask questions like " what is the probability that the coin was heads and the die showed 3".

   When we consider these events, we are implicitly constructing a new probability space whose outcomes are pairs of outcomes from the original experiments. We typically assign probabilities to these outcomes by multiplying probabilities from the original experiments. In this problem we will formalize that procedure.

   Suppose that $(\mathcal{S}_1, \mathcal{P}_1)$ is a probability space (that is, $\mathcal{S}_1$ is a set, and $\mathcal{P}_1$ assigns a probability to each event of $\mathcal{S}_1$, satisfying the Kolmogorov axioms). Suppose also that $(\mathcal{S}_2, \mathcal{P}_2)$ is a probability space. Suppose also that $\mathcal{S}_1$ and $\mathcal{S}_2$ are finite.

   We build a new probability space $(\mathcal{S}, \mathcal{P})$ that models the outcome of both experiments as follows: The outcomes in $\mathcal{S}$ will just be pairs of outcomes of the two given probability spaces:

$$\mathcal{S} = \mathcal{S}_1 \times \mathcal{S}_2 = \{(x, y) \mid x \in \mathcal{S}_1 \text{ and } y \in \mathcal{S}_2\}$$

   We choose the probability of a combined event by multiplying the probabilities of the component events:

$$\mathcal{P}(A) = \sum_{(x,y)\in A} \mathcal{P}_1(\{x\}) \cdot \mathcal{P}_2(\{y\})$$

   (a) Prove that $(\mathcal{S}, \mathcal{P})$ is a probability space.

   (b) Write down the two events "the outcome of the first experiment is x" and "the outcome of the second experiment is y" using set notation. These should be events of the space $(\mathcal{S}, \mathcal{P})$ (in other words, subsets of $\mathcal{S}$).

   (c) Generalizing slightly, if A_1 is an event of the probability space $\mathcal{S}_1$, use set notation to express the event $\hat{A}_1$ representing the fact that A_1 occured in the first experiment. Similarly, if A_2 is an event of $\mathcal{S}_2$, write down the event $\hat{A}_2$ expressing the the fact that A_2 occured in the second experiment.

   (d) Prove that $\hat{A}_1$ and $\hat{A}_2$ from part (c) are independent.