

# Fermat's Little Theorem

## Theorem 11 (Fermat's Little Theorem):

- (a) If  $p$  prime and  $\gcd(p, a) = 1$ , then  $a^{p-1} \equiv 1 \pmod{p}$ .
- (b) For all  $a \in \mathbb{Z}$ ,  $a^p \equiv a \pmod{p}$ .

**Proof.** Let

$$A = \{1, 2, \dots, p-1\}$$

$$B = \{1a \bmod p, 2a \bmod p, \dots, (p-1)a \bmod p\}$$

Claim:  $A = B$ .

- $0 \notin B$ , since  $p \nmid ja$ , so  $B \subset A$ .
- If  $i \neq j$ , then  $ia \bmod p \neq ja \bmod p$ 
  - since  $p \nmid (j-i)a$

Thus  $|B| = p-1$ , so  $A = B$ .

Therefore,

$$\begin{aligned} \prod_{i \in A} i &\equiv \prod_{i \in B} i \pmod{p} \\ \Rightarrow (p-1)! &\equiv a(2a) \cdots (p-1)a = (p-1)! a^{p-1} \pmod{p} \\ \Rightarrow p &\mid (a^{p-1} - 1)(p-1)! \\ \Rightarrow p &\mid (a^{p-1} - 1) \quad [\text{since } \gcd(p, (p-1)!) = 1] \\ \Rightarrow a^{p-1} &\equiv 1 \pmod{p} \end{aligned}$$

It follows that  $a^p \equiv a \pmod{p}$

- This is true even if  $\gcd(p, a) \neq 1$ ; i.e., if  $p \mid a$

Why is this being taught in a CS course?

# Private Key Cryptography

Alice (aka A) wants to send an encrypted message to Bob (aka B).

- A and B might share a private key known only to them.
- The same key serves for encryption and decryption.
- Example: Caesar's cipher  $f(m) = m + 3 \bmod 26$  (shift each letter by three)
  - WKH EXWOHU GLG LW
  - THE BUTLER DID IT

This particular cryptosystem is very easy to solve

- Idea: look for common letters (E, A, T, S)

# One Time Pads

Some private key systems are completely immune to cryptanalysis:

- A and B share the only two copies of a long list of random integers  $s_i$  for  $i = 1, \dots, N$ .
- A sends B the message  $\{m_i\}_{i=1}^n$  encrypted as:

$$c_i = (m_i + s_i) \bmod 26$$

- B decrypts A's message by computing  $c_i - s_i \bmod 26$ .

The good news: bulletproof cryptography

The bad news: horrible for e-commerce

- How do random users exchange the pad?

# Public Key Cryptography

Idea of *public key cryptography* (Diffie-Hellman)

- Everyone's encryption scheme is posted publically
  - e.g. in a "telephone book"
- If A wants to send an encoded message to B, she looks up B's public key (i.e., B's encryption algorithm) in the telephone book
- But only B has the decryption key corresponding to his public key

BIG advantage: A need not know nor trust B.

There seems to be a problem though:

- If we publish the encryption key, won't everyone be able to decrypt?

Key observation: decrypting might be too hard, unless you know the key

- Computing  $f^{-1}$  could be much harder than computing  $f$

Now the problem is to find an appropriate  $(f, f^{-1})$  pair for which this is true

- Number theory to the rescue

# RSA: Key Generation

Generating encryption/decryption keys

- Choose two very large (hundreds of digits) primes  $p, q$ .
  - This is done using probabilistic primality testing
  - Choose a random large number and check if it is prime
  - By the prime number theorem, there are lots of primes out there
- Let  $n = pq$ .
- Choose  $e \in \mathbb{N}$  relatively prime to  $(p - 1)(q - 1)$ . Here's how:
  - Choose  $e_1, e_2$  prime and about  $\sqrt{n}$
  - One must be relatively prime to  $(p - 1)(q - 1)$ 
    - \* Otherwise  $e_1 e_2 \mid (p - 1)(q - 1)$
  - Find out which one using Euclid's algorithm
- Compute  $d$ , the inverse of  $e$  modulo  $(p - 1)(q - 1)$ .
  - Can do this using using Euclidean algorithm
- Publish  $n$  and  $e$  (that's your public key)
- Keep the decryption key  $d$  to yourself.

## RSA: Sending encrypted messages

How does someone send you a message?

- The message is divided into blocks each represented as a number  $M$  between 0 and  $n$ . To encrypt  $M$ , send

$$C = M^e \bmod n.$$

- Need to use fast exponentiation ( $2 \log(n)$  multiplications) to do this efficiently

**Example:** Encrypt “stop” using  $e = 13$  and  $n = 2537$ :

- $s \ t \ o \ p \leftrightarrow 18 \ 19 \ 14 \ 15 \leftrightarrow 1819 \ 1415$
- $1819^{13} \bmod 2537 = 2081$  and  $1415^{13} \bmod 2537 = 2182$  so
- $2081 \ 2182$  is the encrypted message.
- We did not need to know  $p = 43, q = 59$  for that.

# Digital Signatures

How can I send you a message in such a way that you're convinced it came from me (and can convince others).

- Want an analogue of a “certified” signature

Cool observation:

- To sign a message  $M$ , send  $M^d \pmod{n}$ 
  - where  $(n, e)$  is my public key
- Recipient (and anyone else) can compute  $(M^d)^e \equiv M \pmod{n}$ , since  $M$  is public
- No one else could have sent this message, since no one else knows  $d$ .

# Probabilistic Primality Testing

RSA requires really large primes.

- This requires testing numbers for primality.
  - Although there are now polynomial tests, the standard approach now uses probabilistic primality tests

Main idea in probabilistic primality testing algorithm:

- Choose  $b$  between 1 and  $n$  at random
- Apply an easily computable (deterministic) test  $T(b, n)$  such that
  - $T(b, n)$  is true (for all  $b$ ) if  $n$  is prime.
  - If  $n$  is composite, there are lots of  $b$ 's for which  $T(b, n)$  is false

**Example:** Compute  $\gcd(b, n)$ .

- If  $n$  is prime,  $\gcd(b, n) = 1$
- If  $n$  is composite,  $\gcd(b, n) \neq 1$  for some  $b$ 's
  - Problem: there may not be that many witnesses



**Example:** Compute  $b^{n-1} \bmod n$

- If  $n$  is prime  $b^{n-1} \equiv 1 \pmod{n}$  (Fermat)
- Unfortunately, there are some composite numbers  $n$  such that  $b^{n-1} \equiv 1 \pmod{n}$ 
  - These are called *Carmichael numbers*

There are tests  $T(b, n)$  with the property that

- $T(b, n) = 1$  for all  $b$  if  $n$  is prime
- $T(b, n) = 0$  for at least  $1/3$  of the  $b$ 's if  $n$  is composite
- $T(b, n)$  is computable quickly (in polynomial time)

Constructing  $T$  requires a little more number theory

- Beyond the scope of this course.

Given such a test  $T$ , it's easy to construct a probabilistic primality test:

- Choose 100 (or 200)  $b$ 's at random
- Test  $T(b, n)$  for each one
- If  $T(b, n) = 0$  for any  $b$ , declare  $b$  composite
  - This is definitely correct
- If  $T(b, n) = 1$  for all  $b$ 's you chose, declare  $n$  prime
  - This is highly likely to be correct

# Security is Subtle

There are lots of ways of “misapplying” RSA, even assuming that factoring is hard.

- The public key  $n = pq$ , the product of two large primes
- How do you find the primes?
  - Guess a big odd number  $n_1$ , check if it's prime
  - If not, try  $n_1 + 2$ , then  $n_1 + 4$ , ...
  - Within roughly  $\log(n_1)$  steps, you should find a prime;
- How do you find the second prime?
  - Guess a big odd number  $n_2$ , check if it's prime
  - ...
- Suppose, instead, you started with the first prime (call it  $p$ ), and checked  $p + 2$ ,  $p + 4$ ,  $p + 6$ , ..., until you found another prime  $q$ , and used that.
  - Is that a good idea? NO!!!

If  $n = pq$ , then  $p$  is the first prime less than  $\sqrt{n}$ , and  $q$  is the first prime greater than  $\sqrt{n}$ .

- You can find both easily!

## More to Explore

If you like number theory, consider taking

- MATH 332: Algebra and Number Theory

If you're interested in cryptography, try

- CS 487: Introduction to Cryptography

For a brief introduction to some current number theory, check out

<http://math.arizona.edu/~mcleman/CoolNumbers/CoolNumbers.html>

- The Ten Coolest Numbers
- thanks to Rob Tirrell for pointing this out

# Combinatorics

Problem: How to count without counting.

- How do you figure out how many things there are with a certain property without actually enumerating all of them.

Sometimes this requires a lot of cleverness and deep mathematical insights.

But there are some standard techniques.

- That's what we'll be studying.

## Sum and Product Rules

**Example 1:** In New Hampshire, license plates consisted of two letters followed by 3 digits. How many possible license plates are there?

**Answer:** 26 choices for the first letter, 26 for the second, 10 choices for the first number, the second number, and the third number:

$$26^2 \times 10^3 = 676,000$$

**Example 2:** A traveling salesman wants to do a tour of all 50 state capitals. How many ways can he do this?

**Answer:** 50 choices for the first place to visit, 49 for the second, ...:  $50!$  altogether.

Chapter 4 gives general techniques for solving counting problems like this. Two of the most important are:

**The Sum Rule:** If there are  $n(A)$  ways to do  $A$  and, distinct from them,  $n(B)$  ways to do  $B$ , then the number of ways to do  $A$  *or*  $B$  is  $n(A) + n(B)$ .

- This rule generalizes: there are  $n(A) + n(B) + n(C)$  ways to do  $A$  *or*  $B$  *or*  $C$
- In Section 4.8, we'll see what happens if the ways of doing  $A$  and  $B$  aren't distinct.

**The Product Rule:** If there are  $n(A)$  ways to do  $A$  and  $n(B)$  ways to do  $B$ , then the number of ways to do  $A$  *and*  $B$  is  $n(A) \times n(B)$ . This is true if the number of ways of doing  $A$  and  $B$  are independent; the number of choices for doing  $B$  is the same regardless of which choice you made for  $A$ .

- Again, this generalizes. There are  $n(A) \times n(B) \times n(C)$  ways to do  $A$  *and*  $B$  *and*  $C$

## Some Subtler Examples

**Example 3:** If there are  $n$  Senators on a committee, in how many ways can a subcommittee be formed?

Two approaches:

1. Let  $N_1$  be the number of subcommittees with 1 senator ( $n$ ),  $N_2$  the number of subcommittees with 2 senator ( $n(n-1)/2$ ),  $\dots$

According to the sum rule:

$$N = N_1 + N_2 + \dots + N_n$$

- It turns out that  $N_k = \frac{n!}{k!(n-k)!}$  ( $n$  choose  $k$ ); this is discussed in Section 4.4
- A subtlety: What about  $N_0$ ? Do we allow subcommittees of size 0? How about size  $n$ ?
  - The problem is somewhat ambiguous.

If we allow subcommittees of size 0 and  $n$ , then there are  $2^n$  subcommittees altogether.

- This is the same as the number of subsets of the set of  $n$  Senators: there is a 1-1 correspondence between subsets and subcommittees.

2. Simpler method: Use the product rule!

- Each senator is either in the subcommittee or out of it: 2 possibilities for each senator:
  - $2 \times 2 \times \cdots \times 2 = 2^n$  choices altogether

General moral: In many combinatorial problems, there's more than one way to analyze the problem.



How many ways can the full committee be split into two sides on an issue?

This question is also ambiguous.

- If we care about which way each Senator voted, then the answer is again  $2^n$ : Each subcommittee defines a split + vote (those in the subcommittee vote Yes, those out vote No); and each split + vote defines a subcommittee.
- If we don't care about which way each Senator voted, the answer is  $2^n/2 = 2^{n-1}$ .
  - This is an instance of the Division Rule.

# Coping with Ambiguity

If you think a problem is ambiguous:

1. Explain why
2. Choose one way of resolving the ambiguity
3. Solve the problem according to your interpretation
  - Make sure that your interpretation doesn't render the problem totally trivial

## More Examples

**Example 4:** How many legal configurations are there in Towers of Hanoi with  $n$  rings?

**Answer:** The product rule again: Each ring gets to “vote” for which pole it’s on.

- Once you’ve decided which rings are on each pole, their order is determined.
- The total number of configurations is  $3^n$

**Example 5:** How many distinguishable ways can the letters of “computer” be arranged? How about “discrete”?

For computer, it’s  $8!$ :

- 8 choices for the first letter, for the second, ...

Is it  $8!$  for discrete? Not quite.

- There are two e’s

Suppose we called them  $e_1, e_2$ :

- There are two “versions” of each arrangement, depending on which e comes first:  $\text{discre}_1\text{te}_2$  is the same as  $\text{discre}_2\text{te}_1$ .
- Thus, the right answer is  $8!/2!$

**Division Rule:** If there is a  $k$ -to-1 correspondence between objects of type  $A$  with objects of type  $B$ , and there are  $n(A)$  objects of type  $A$ , then there are  $n(A)/k$  objects of type  $B$ .

A  *$k$ -to-1 correspondence* is an onto mapping in which every  $B$  object is the image of exactly  $k$   $A$  objects.

# Permutations

A *permutation* of  $n$  things taken  $r$  at a time, written  $P(n, r)$ , is an arrangement in a row of  $r$  things, taken from a set of  $n$  distinct things. Order matters.

**Example 6:** How many permutations are there of 5 things taken 3 at a time?

**Answer:** 5 choices for the first thing, 4 for the second, 3 for the third:  $5 \times 4 \times 3 = 60$ .

- If the 5 things are  $a, b, c, d, e$ , some possible permutations are:

$abc \quad abd \quad abe \quad acb \quad acd \quad ace$   
 $adb \quad adc \quad ade \quad aeb \quad aec \quad aed$   
 $\dots$

In general

$$P(n, r) = \frac{n!}{(n-r)!} = n(n-1) \cdots (n-r+1)$$

# Combinations

A *combination* of  $n$  things taken  $r$  at a time, written  $C(n, r)$  or  $\binom{n}{r}$  (“ $n$  choose  $r$ ”) is any subset of  $r$  things from  $n$  things. Order makes no difference.

**Example 7:** How many ways are there of choosing 3 things from 5?

**Answer:** If order mattered, then it would be  $5 \times 4 \times 3$ . Since order doesn’t matter,

$abc, acb, bac, bca, cab, cba$

are all the same.

- For way of choosing three elements, there are  $3! = 6$  ways of ordering them.

Therefore, the right answer is  $(5 \times 4 \times 3)/3! = 10$ :

$abc \quad abd \quad abe \quad acd \quad ace$   
 $ade \quad bcd \quad bce \quad bde \quad cde$

In general

$$C(n, r) = \frac{n!}{(n-r)!r!} = n(n-1) \cdots (n-r+1)/r!$$

## More Examples

**Example 8:** How many full houses are there in poker?

- A full house has 5 cards, 3 of one kind and 2 of another.
- E.g.: 3 5's and 2 K's.

**Answer:** You need to find a systematic way of counting:

- Choose the denomination for which you have three of a kind: 13 choices.
- Choose the three:  $C(4, 3) = 4$  choices
- Choose the denomination for which you have two of a kind: 12 choices
- Choose the two:  $C(4, 2) = 6$  choices.

Altogether, there are:

$$13 \times 4 \times 12 \times 6 = 3744 \text{ choices}$$

# 0!

It's useful to define  $0! = 1$ .

Why?

1. Then we can inductively define

$$(n+1)! = (n+1)n!,$$

and this definition works even taking 0 as the base case instead of 1.

2. A better reason: Things work out right for  $P(n, 0)$  and  $C(n, 0)$ !

How many permutations of  $n$  things from  $n$  are there?

$$P(n, n) = \frac{n!}{(n-n)!} = \frac{n!}{0!} = n!$$

How many ways are there of choosing  $n$  out of  $n$ ?  
0 out of  $n$ ?

$$\binom{n}{n} = \frac{n!}{n!0!} = 1$$
$$\binom{n}{0} = \frac{n!}{0!n!} = 1$$