CS280, Spring 2004: Final

1. [4 points] Which of the following relations on $\{0,1,2,3\}$ is an equivalence relation. (If it is, explain why. If it isn't, explain why not.) Just saying "Yes" or "No" with no explanation gets 0 points. The explanations can be very short though.)

   (a) $\{(0,0,),(1,1),(2,2),(3,3)\}$

   (b) $\{(0,0),(0,1),(1,0),(1,1)\}$.

   Solution: (a) is an equivalence relation, since it is reflexive, symmetric, and transitive. (b) is not an equivalence relation, since it is not reflexive; neither (2,2) nor (3,3) are in the relation. (It is symmetric and transitive; you got one point if you said that, even if you didn't get the right answer.)

   Grading: Basically, 1 point for the correct answer and 1 point for the reason. For (b), 1 point was given for the wrong answer if you said it was symmetric and transitive.

2. [5 points] Recall that a composite n is a Carmichael number if for any b relatively prime to n, $b^{n-1} \equiv 1 \pmod{n}$. Show that 1105 is a Carmichael number. [Hint: $1105 = 5 \cdot 13 \cdot 17$.]

   Solution: We want to show that if b is relatively prime to 1105, then $b^{1104} \equiv 1 \pmod{1105}$. By the Chinese Remainder Theorem, it suffices to show that $b^{1104} \equiv 1 \pmod{5}$, $b^{1104} \equiv 1 \pmod{13}$, and $b^{1104} \equiv 1 \pmod{17}$. By Fermat's Little Theorem, $a^{p-1} \equiv 1 \pmod{p}$ if a is relatively prime to p. By assumption, b is relatively prime to 1105, so b is also relatively prime to 5, 13, and 17. Thus $b^4 \equiv 1 \pmod{5}$, $b^{12} \equiv 1 \pmod{13}$, and $b^{16} \equiv 1 \pmod{17}$. Thus, $b^{4n} \equiv 1 \pmod{5}$ for all n; similarly $b^{12n} \equiv 1 \pmod{13}$ and $b^{16n} \equiv 1 \pmod{17}$ for all n. Since 4, 12, and 16 all divide 1104 evenly, it follows that $b^{1104} \equiv 1 \pmod{5}$, $b^{1104} \equiv 1 \pmod{13}$, and $b^{1104} \equiv 1 \pmod{17}$.

   Grading: You lost one point if you didn't explicitly state that since b is relatively prime to 1105, it is relatively prime to each of 5, 13, and 17. You also lost one point if you didn't justify the fact that $b^{p-1} \equiv 1 \pmod{p}$ for each of $p = 5, 13, 17$ (i.e., you had to mention Fermat's Little Theorem). You also lost one point for not mentioning the Chinese Remainder Theorem.

3. [4 points] Consider any six natural numbers $n_1, \ldots, n_6$. Show that the sum of some subsequence of consecutive numbers is divisible by 6 (e.g., perhaps $n_3+n_4+n_5$ is divisible by 6, or n_4 itself is divisible by 6, or n_2+n_3 is divisible by 6). [Hint: Look at the sums 0, n_1, n_1+n_2, $n_1+n_2+n_3$, ..., $n_1+n_2+\cdots+n_6$, and think in terms of mod 6.]

   Solution: Following the hint, consider each of the sums 0, n_1, n_1+n_2, $n_1+n_2+n_3$, ..., $n_1+n_2+\cdots+n_6$ mod 6. If any of them is 0 mod 6, we're done (since that sum is divisible by 6). By the pigeonhole principle, two of the sums must be the same mod 6. (The "pigeons" are the sums, the "holes" are 1, ..., 5, since all the sums must be in the range 1, ..., 5 mod 6.) If two sums are the same mod 6, their difference must be 0 mod 6, hence must be divisible by 6. But the difference between two such sequences is a sum of consecutive integers. (The difference between $n_1+\cdots+n_k$ and $n_1+\cdots+n_m$ (where $k > m$), is $n_{m+1}+\cdots+n_k$.) Thus, this sum of consecutive integers is divisible by 6.

Grading: This one was hard for most people. You got 1 point for saying "pigeon hole"; 1 point for mentioning the trivial case where one of $n_1, \ldots, n_6 \equiv 0 \pmod 6$; 1 point for saying that two of the sequences from n_1, $n_1 + n_2$, $\ldots n_1 + \cdots + n_6$ must be congruent mod 6; and 1 point for saying their difference must be congruent to 0 mod 6. No points were given for showing that the sum of 6 consecutive numbers is always divisible by 6. No points were given for listing pairs, triplets, etc of numbers that could not be used.

4. [5 points] What is the coefficient of x^{25} in the binomial expansion of $(2x - \frac{3}{x^2})^{58}$? (There's no need to simplify the expression.)

   Solution: By the Binomial Theorem,

$$(2x - \frac{3}{x^2})^{58} = \sum_{k=0}^{58} \binom{58}{k} (2x)^k (-\frac{3}{x^2})^{58-k} = \sum_{k=1}^{58} \binom{58}{k} 2^k (-3)^{58-k} x^{k-2(58-k)}.$$

   $k - 2(58-k) = 25$ iff $3k - 116 = 25$ iff $k = 47$. Thus, the coefficient of x^{25} is $\binom{58}{47} 2^{47} (-3)^{11}$.

   Grading: 2 points for using the binomial expansion; 2 points for getting $k = 47$ or $k = 11$; 1 point for getting the right answer.

5. [5 points] Prove that $3^n \geq n^3$ for all $n \geq 3$.

   Solution: We proceed by induction. Let $P(n)$ be the statement that $3^n \geq n^3$. $P(3)$ is obviously true, since $3^3 = 3^3$. Suppose that $P(k)$ is true for $k \geq 3$. That is, assume that $3^k \geq k^3$. We want to show that $3^{k+1} > (k+1)^3$. By the Binomial Theorem, $(k+1)^3 = k^3 + 3k^2 + 3k + 1$. Since $k \geq 3$, it follows that

   - $3k^2 \leq kk^2 = k^3$
   - $3k + 1 \leq 3k + k = 4k \leq 9k \leq k^2 k = k^3$

   Thus, $(k+1)^3 \leq k^3 + 3k^2 + 3k + 1 \leq 3k^3$. By the induction hypothesis, $k^3 \leq 3^k$, so $(k+1)^3 l33^k = 3^{k+1}$. This completes the induction proof.

   Grading: Here we used the standard induction grading scheme. 1 point for stating the induction hypothesis; 1 point for the base case; 1 point for the conclusion; 2 points for an appropriate use of the inductive hypothesis.

6. [3 points] How many 5-card hands have exactly 3 kings?

   Solution: There are $\binom{4}{3}$ ways of choosing the kings and $\binom{48}{2}$ ways of choosing the remaining two cards, so there are

$$\binom{4}{3}\binom{48}{2} = 4 \times 48 \times 47/2 = 96 \times 47$$

   ways altogether. (See the next problem for grading comments.)

7. [4 points] A committee of 7 is to be chosen from 8 men and 9 women. How many contain either Alice or Bob, but not both? [You do *not* have to simplify the expression that you get.]

Solution: There are $\binom{16}{6}$ committees containing Alice (you have to choose 6 committee memebers from the remaining 16 people). Similarly, there are $\binom{16}{6}$ committees containing Bob, and $\binom{15}{5}$ committees containing both Bob and Alice. Applying the inclusion-exclusion rule, there are $2\times\binom{16}{6}-\binom{15}{5}$ containing either Alice or Bob, and $2\times\binom{16}{6}-2\times\binom{15}{5}$ containing either Alice or Bob, but not both.

Grading: If you used the correct method, but made a single small mistake (like forgetting to subtract one from the pool of potential candidates when computing the number of committes that contain Alice, or forgetting to subtract $\binom{15}{5}$ twice by the inclusion-exclusion rule) one point was deducted. If the attempt was somewhat correct but had multiple minor mistakes, two or three points were deducted. (The same comments apply to the previous problem.)

8. [4 points] There are N different types of coupons. Each time a coupon collector obtains a coupon it is equally likely to be any one of those. What is the probability that the collector needs more than k ($k > N$) coupons to complete his collection (have at least one of each type of coupon)? (You do *not* have to simplify the expression that you get.) [Hint: In other words, what is the probability that at least one type coupon is missing among the first k coupons.]

   Solution: Let C_i be the event that the coupon collector doesn't have coupon i, for $i = 1, \dots, N$. It's easy to see that $\Pr(C_i) = (\frac{N-1}{N})^k$. We're interested in $\Pr(C_1 \vee \dots \vee C_N)$. We'll use the inclusion-exclusion rule to compute this. First note that $\Pr(C_{i_1} \cap \dots \cap C_{i_m}) = (\frac{N-m}{N})^k$ (where $i_1, \dots, i_m$ are all distinct); that's the probability that the coupon collector doesn't have any of coupons $i_1, \dots, i_m$. Now by the inclusion-exclusion rule

$$\Pr(C_1 \vee \dots \vee C_N) = \sum_{m=1}^{N} \binom{N}{m} (-1)^{m+1} (\frac{N-m}{N})^k.$$

   [If $k < N$, then the probability that the ticket collector is missing a coupon after k steps is 1. This expression is actually correct even in that case.]

   Grading: This was actually graded out of 4, not 5, so the exam was out of 69, not 70. This helped almost everyone, since most people did badly on this problem. You got partial credit if you used Markov's inequality (typically 1 or 2 points, depending on what you did).

9. [10 points] For each of the following clearly answer yes or no (no justification is needed nor will it help you). If you answer correctly you get +1 but if you give the wrong answer you get -1 points. A skipped item earns 0 points. In any case the overall grade for this particular problem will not be less than zero.

   (a) If A and B are disjoint events then they are independent.
   Solution: False. If A and B are disjoint, then $A \cap B = \emptyset$, so $\Pr(A \cap B) = 0$. If $\Pr(A) > 0$ and $\Pr(B) > 0$, then $\Pr(A \cap B) \neq \Pr(A)\Pr(B)$.

   (b) If A and B are independent events then so are $\overline{A}$ and B.
   Solution: True. Since A and B are independent,

$$\Pr(\overline{A} \cap B) = \Pr(B) - \Pr(A \cap B) = \Pr(B) - \Pr(A)\Pr(B) = \Pr(B)(1 - \Pr(A) = \Pr(B)\Pr(\overline{A}).$$

(c) A sum of Bernoulli random variables is a binomial random variable.

Solution: False. For example, take $X + X$, where X is Bernoulli. This is not a binomial random variable. This actually is true if all the Bernoulli random variables are iid (independent and identically distributed).

(d) If X and Y are independent then $V(X + Y) = V(X) + V(Y)$.

Solution: True. This was done in class, but here's a formal proof:

$$\begin{aligned} & V(X+Y) = E((X+Y)^2) - E(X+Y)^2 = E(X^2 + 2XY + Y^2) - (E(X)^2 + 2E(X)E(Y) - E \\ = \; & (E(X^2) - E(X)^2) + (E(Y^2) - E(Y)^2) + 2E(XY) - E(X)E(Y) = V(X) + V(Y). \end{aligned}$$

Here I'm using the fact that if X and Y are independent, then $E(XY) = E(X)E(Y)$.

(e) If a is a real number and X is a random variable, then $V(aX) = aV(X)$.

Solution: False. $V(aX) = a^2V(X)$.
(Proof: $V(aX) = E((aX)^2) - (E(aX))^2 = a^2E(X^2) - a^2E(X)^2 = a^2V(X)$.)

(f) If X and Y are *not* independent then $E(X + Y) \neq E(X) + E(Y)$.

Solution: False. $E(X + Y) = E(X) + E(Y)$ whether or not XR and Y are independent.

(g) If A and B are independent events then $\Pr(A \cup B) = \Pr(A) + \Pr(B)$.

Solution: False. $\Pr(A \cup B) = \Pr(A) + \Pr(B) - \Pr(A \cap B)$. If A and B are independent, then $\Pr(A \cap B) = \Pr(A)\Pr(B)$, but it does not equal 0 in general.

(h) Suppose that you have a coin that has probability .2 of landing heads, and you toss it 100 times. Let X be the number of times that the coin lands heads. Then $Pr(X \geq 60) \leq 1/100$.

Solution: True. (This is Chebyshev's inequality, using the fact that X is a binomial random variable, with $p = .2$ and $n = 100$. Thus, $E(X) = 20$, $Var(X) = np(1-p) = 16$, and $\sigma_X = 4$. Note that $\Pr(X \geq 60) = \Pr(|X - 20| \geq 40)$, and $40 = 10\sigma_X$. Thus, $\Pr(X \geq 60) \leq 1/100$.)

(i) Let T_n be the number of times a fair coin lands heads after being flipped n times. Then

$$\lim_{n\to\infty} \Pr\left(\left|\frac{T_n}{n} - \frac{1}{2}\right| < .01\right) = 1.$$

Solution: True. The Law of Large numbers says that for all ϵ,

$$\lim_{n\to\infty} \Pr\left(\left|\frac{T_n}{n} - \frac{1}{2}\right| < \epsilon\right) = 1.$$

(j) Taking T_n as above,

$$\lim_{n\to\infty} \Pr\left(\left|\frac{2T_n - n}{\sqrt{n}}\right| < 1\right) = 1.$$

Solution: False. By the Central Limit Theorem, the required limit is the area under the curve of the Gaussian distribution, between -1 and 1. Although it is close to 1, it is not 1.

10. [4 points] There are three cards. The first is red on both sides; the second is black on both sides; and the third is red on one side and black on the other side. A card is randomly selected and randomly placed on the table. The color that we see is red. What is the probability that the hidden side is black?

    Solution: Let the three cards be a, b, and c. Let $a1$ denote the first side of the first card, $a2$ the second side, and so on. So the event that you see red is $R = \{a1, a2, c1\}$. In only one of these cases is the other side black, so the probability that the other side is black is 1/3. (More formally, the event that the other side is black is $OB = \{b1, b2, c1\}$, so we are interested in $\Pr(OB \mid R) = \Pr(OB \cap R)/\Pr(R) = 1/3$.

    Grading: You lost a point if you got the answer 2/3 rather than 1/3, if this was the result of using 1/3 in the numerator rather than 1/6. You lost two points if you got 1/2 because you didn't try to restrict the sample space.

11. [4 points] Two fair dice are rolled. What is the probability at least one lands on 6 given that the dice land on different numbers?

    Solution: The event D that the dice land on different numbers consist of the 30 pairs (i, j) such that $i \neq j$. The event S that one die lands 6 consists of the 11 events of the form $(6, j)$ and $(i, 6)$. (There are only 11 since you don't want to double-count $(6, 6)$.) Note that $S \cap D$ has 10 elements (all the elements in S except for (6,6).) We are interested in $\Pr(S \mid R) = 10/30 = 1/3$.

12. [5 points]

    Input: n (a positive integer)
    $\mathit{factorial} \leftarrow 1$
    $i \leftarrow 1$
    while $i < n$ **do**
        $i \leftarrow i + 1$
        $\mathit{factorial} \leftarrow i * \mathit{factorial}$

    Prove that the program terminates with $\mathit{factorial} = n!$ given input n, using an appropriate loop invariant.

    Solution: The loop invariant is that before the kth iteration, $\mathit{factorial} = k!$ and $i = k$. We prove this by induction, taking $P(k)$ to be the statement of the loop invariant. The base case is obvious, since initially $\mathit{factorial} = 1$ and $i = 1$. Suppose that $P(k)$ holds; we prove $P(k+1)$. Clearly during the $(k+1)$st interation, i is set to $i+1$, which is $k+1$, given the inductive hypothesis, and *factorial* is then set to $i * \mathit{factorial}$, which is $(k+1)k! = (k+1)!$. This completes the inductive proof. It follows that before the nth iteration, $i = n$ and $\mathit{factorial} = n!$. The program then terminates, since it is not the case that $i < n$.

    Grading: It's fine to say that the loop invariant is $\mathit{factorial} = i!$, but then you have to be really careful to state your inductive hypothesis. Remember that if $P(k)$ is your inductive hypothesis, it should have a k in it. Saying $P(k)$ is $\mathit{factorial} = i!$ lost you one point. (What's $P(1)$ in that case? What about $P(n)$?) You also lost one point if you didn't talk about termination. Finally, a lot of you took the inductive hypothesis $P(n)$ to be "if the input is n, then the program halts with $\mathit{factorial} = n!$. While that's a perfectly good inductive hypothesis, it's very hard to prove the inductive step, since when the input is $k+1$ you have to carefully reduce it to the case where the input is k. Most people lost 2 points if they tried to do this argument.

13. [7 points] Translate the following argument into propositional logic, then determine whether it is valid. (Remember that an argument is valid if, whenever the premises are true, the conclusion is true.)

    If I like mathematics, then I will study.
    Either I don't study or I pass mathematics
    If I don't graduate, then I didn't pass mathematics

    If I like mathematics, then I will graduate

    [3 points for the translation, and 4 points for proving that it is valid or not valid.]

    Solution: Take L to be "I like mathematics", S to be "I will study", P to be "I pass mathematics", and G to be "I will graduate", the argument becomes

    $L \Rightarrow S$
    $\neg S \vee P$ (we accepted xor instead of $\vee$ here; both give the same answer)
    $\neg G \Rightarrow \neg P$

    $L \Rightarrow G$

    This argument is valid. To see this, let T be any truth assignment that makes the premises true. That is, $T(L \Rightarrow S) = T(\neg S \vee P) = T(\neg G \Rightarrow \neg P) = \mathit{true}$. We need to show that T makes the conclusion true. That is, we want to show that $T(L \Rightarrow G) = \mathit{true}$. If $T(L) = \mathit{false}$, then $T(L \Rightarrow G) = \mathit{true}$, so we're done. If $T(L) = \mathit{true}$, then $T(S) = \mathit{true}$ (since $T(L \Rightarrow S) = \mathit{true}$). It then follows that $T(P) = \mathit{true}$ (since $T(\neg S \vee P) = \mathit{true}$, and the only way that the disjunction can be true is if either $T(\neg S) = \mathit{true}$ or $T(P) = \mathit{true}$). It follows from that that $T(G) = \mathit{true}$, since $T(\neg G \Rightarrow \neg P) = \mathit{true}$. (If $T(G) = \mathit{false}$, then $T(\neg G) = \mathit{true}$, which, combined with $T(P) = \mathit{true}$, means that $T(\neg G \Rightarrow \neg P) = \mathit{false}$. What we've just shown is that if $T(L) = \mathit{true}$ then $T(G) = \mathit{true}$. It follows that $T(L \Rightarrow G) = \mathit{true}$. That is, any truth assignment that makes all the premises true also makes the conclusion true.

    Grading: For checking the the argument was valid, if you gave a correct proof for your translation, this was accepted, even if the translation was wrong. Some people seemed to misunderstand what it meant for an argument to be valid, despite the reminder given in the question.

14. [5 points] Suppose that $K(x, y)$ "x knows (i.e., is acquainted with) y". Translate each of the following sentences into first-order logic:

(a) Nobody knows Alice.

(b) Sam knows everyone.

(c) There is someone that Sam doesn't know.

(d) Everyone knows someone.

(e) Sam knows everyone that David knows.

Solution:

(a) $\forall x \neg K(x, Alice)$ (or $\neg \exists x K(x, Alice)$).

(b) $\forall x K(Sam, x)$.

(c) $\exists x \neg K(Sam, x)$.

(d) $\forall x \exists y K(x, y)$.

(e) $\forall x (K(David, x) \Rightarrow K(Sam, x))$.

Grading: Each part got one point. Half a point was taken off for mixing up existential and universal quantifiers, if this mistake was made more than once; we tried not to penalize you for it each time. In general, the biggest mistake was people putting quantifiers inside predicates. (It's not OK to write $K(x, \exists y)$, for example.)