CS 2800 – Fall 2009
Review Problems for the Final

1. Convert the following statements into propositional logic and determine if the given argument is sound:

   Rainy days make gardens grow.
   Gardens don't grow if it is not hot.
   It always rains on a day that is not hot.
   Therefore, if it is not hot, then it is hot.

   [**Ans**] Let R denote a day is rainy, G denote gardens grow and H denote it is hot. We then have the premises $R \rightarrow G$, $\neg H \rightarrow \neg G$ and $\neg H \rightarrow R$. We want to show that $\neg H \rightarrow H$. Taking the contrapositive of the second premise and applying the hypothetical syllogism inference rule twice gives us the conclusion. Hence the argument is sound.

2. We define the set $S = \{x | x \notin S\}$. Does $S \in S$? Does $S \notin S$?
   [**Ans**] Assuming either statement is true instantly yields a contradiction; the set S is thus poorly defined.

3. Show that if A and B are countable sets, then $A \times B$ is also countable.
   [**Ans**] Either use the fact that we have a countable union of countable sets or the table traversal trick as seen in the proof that the rational numbers are countable.

4. Show that the power set of the set of positive integers $\mathbb{Z}^+$ is uncountable.
   [**Ans**] Set up a one-to-one correspondence between the elements of the power set of $\mathbb{Z}^+$ and the set of infinitely long binary strings; if the integer i appears in a subset then the i^{th} most significant bit of the corresponding binary string is set (i.e. a 1) and otherwise it is a 0. Since the set of infinitely long binary strings is uncountable (by diagonalization argument), the power set of the $\mathbb{Z}^+$ is uncountable as well.

5. What is the best big-O function for

   (a) $n^3 + \sin n^7$ [**Ans**] $O(n^3)$

   (b) $(x + 2)\log_2(x^2 + 1) + \log_2(x^3 + 1)$ [**Ans**] $O(x \log x)$

6. What is the worst-case time complexity of the following algorithms?

   (a) One that prints out all the ways to place the numbers $1, 2, \ldots, n$ in a row.
   [**Ans**] $O(n!)$

   (b) One that enumerates all the subsets of a given finite set of size n.
   [**Ans**] $O(2^n)$

7. Prove that if n is an integer that is not a multiple of 3, then $n^2 \equiv 1 \pmod 3$.
[**Ans**] If n is not a multiple of 3, then it can be written as $3k+1$ or $3k+2$ where k is an integer. Argue that in both cases, $n^2 \equiv 1 \pmod 3$.

8. Explain in words the difference between $a|b$ and $\frac{b}{a}$.
[**Ans**] $a|b$ is a proposition; $\frac{b}{a}$ is a number

9. Find an inverse of 17 modulo 19.
[**Ans**] 9

10. Show using induction that $11|(10^{2k+1}+1)$ for all non-negative integers k.

11. Show using induction that for all positive integers n:

$$1 - 2 + 2^2 - 2^3 + \ldots + (-1)^n 2^n = \frac{2^{n+1}(-1)^n + 1}{3}$$

12. How many strings of eight English letters are there
    (a) if letters can be repeated? [**Ans**] 26^8
    (b) if no letter can be repeated? [**Ans**] $P(26, 8)$
    (c) that start with X, if letters can be repeated? [**Ans**] 26^7
    (d) that start with X, if no letter can be repeated? [**Ans**] $P(25, 7)$
    (e) that start and end with X, if letters can be repeated? [**Ans**] 26^6
    (f) that start with the letters BO (in that order), if letters can be repeated? [**Ans**] 26^6
    (g) that start and end with the letters BO (in that order), if letters can be repeated? [**Ans**] 26^4
    (h) that start or end with the letters BO (in that order), if letters can be repeated? [**Ans**] $2 \cdot 26^6 - 26^4$

13. Show that there are at least 6 people in California (population 36 million) with the same three initials who were born on the same day of the year (but not necessarily the same year). Assume everyone has three initials.
[**Ans**] Pigeon-hole principle — at least $\lceil \frac{36 \times 10^6}{26^3 \cdot 365} \rceil = 6$ end up in the same bin.

14. How many different strings can be made from the letters in *ORONO* using some or all the letters?
[**Ans**] 63

15. What is the probability that exactly four heads appear when a fair coin is flipped five times, given that the first flip came up heads?
[**Ans**] $\binom{4}{3}(\frac{1}{2})^3\frac{1}{2}$

16. In the following question, an experiment consists of picking a bit string of length 5 at random. Consider the following events:

- E_1: the bit string chosen begins with a 1.
- E_2: the bit string chosen ends with a 1.
- E_3: the bit string chosen has exactly three 1s.

(a) Find $p(E_1)$, $p(E_2)$ and $p(E_3)$. **[Ans]** $\frac{1}{2}, \frac{1}{2}, \frac{5}{16}$

(b) Find $p(E_1|E_3)$. **[Ans]** $\frac{3}{5}$

(c) Find $p(E_2|E_1 \cap E_3)$. **[Ans]** $\frac{1}{2}$

(d) Are E_1 and E_2 independent? Justify your answer. **[Ans]** Yes

(e) Are E_2 and E_3 independent? Justify your answer. **[Ans]** No