Permutations

A *permutation* of n things taken r at a time, written $P(n, r)$, is an arrangement in a row of r things, taken from a set of n distinct things. Order matters.

Example 6: How many permutations are there of 5 things taken 3 at a time?

Answer: 5 choices for the first thing, 4 for the second, 3 for the third: $5 \times 4 \times 3 = 60$.

- If the 5 things are a, b, c, d, e, some possible permutations are:

$$\begin{array}{llllll} abc & abd & abe & acb & acd & ace \\ adb & adc & ade & aeb & aec & aed \\ \ldots \end{array}$$

In general

$$P(n, r) = \frac{n!}{(n-r)!} = n(n-1)\cdots(n-r+1)$$

Combinations

A *combination* of n things taken r at a time, written $C(n,r)$ or $\binom{n}{r}$ ("n choose r") is any subset of r things from n things. Order makes no difference.

Example 7: How many ways are there of choosing 3 things from 5?

Answer: If order mattered, then it would be $5 \times 4 \times 3$. Since order doesn't matter,

$$abc, acb, bac, bca, cab, cba$$

are all the same.

- For way of choosing three elements, there are $3! = 6$ ways of ordering them.

Therefore, the right answer is $(5 \times 4 \times 3)/3! = 10$:

$$\begin{array}{ccccc} abc & abd & abe & acd & ace \\ ade & bcd & bce & bde & cde \end{array}$$

In general

$$C(n,r) = \frac{n!}{(n-r)!r!} = n(n-1)\cdots(n-r+1)/r!$$

More Examples

Example 8: How many full houses are there in poker?

- A full house has 5 cards, 3 of one kind and 2 of another.
- E.g.: 3 5's and 2 K's.

Answer: You need to find a systematic way of counting:

- Choose the denomination for which you have three of a kind: 13 choices.
- Choose the three: $C(4,3) = 4$ choices
- Choose the denomination for which you have two of a kind: 12 choices
- Choose the two: $C(4,2) = 6$ choices.

Altogether, there are:

$$13 \times 4 \times 12 \times 6 = 3744 \text{ choices}$$

0!

It's useful to define $0! = 1$.

Why?

1. Then we can inductively define

$$(n+1)! = (n+1)n!,$$

   and this definition works even taking 0 as the base case instead of 1.

2. A better reason: Things work out right for $P(n,0)$ and $C(n,0)$!

How many permutations of n things from n are there?

$$P(n,n) = \frac{n!}{(n-n)!} = \frac{n!}{0!} = n!$$

How many ways are there of choosing n out of n?
0 out of n?

$$\binom{n}{n} = \frac{n!}{n!0!} = 1$$

$$\binom{n}{0} = \frac{n!}{0!n!} = 1$$

More Questions

Q: How many ways are there of choosing k things from $\{1, \ldots, n\}$ if 1 and 2 can't both be chosen? (Suppose $n, k \geq 2$.)

A: First find all the ways of choosing k things from n—$C(n, k)$. Then subtract the number of those ways in which both 1 and 2 are chosen:

- This amounts to choosing $k-2$ things from $\{3, \ldots, n\}$: $C(n-2, k-2)$.

Thus, the answer is

$$C(n, k) - C(n-2, k-2)$$

Q: What if order matters?

A: Have to compute how many ways there are of picking k things, two of which are 1 and 2.

$$P(n, k) - k(k-1)P(n-2, k-2)$$

Q: How many ways are there to distribute four distinct balls evenly between two distinct boxes (two balls go in each box)?

A: All you need to decide is which balls go in the first box.

$$C(4, 2) = 6$$

Q: What if the boxes are indistinguishable?

A: $C(4, 2)/2 = 3$.

Combinatorial Identities

There all lots of identities that you can form using $C(n, k)$. They seem mysterious at first, but there's usually a good reason for them.

Theorem 1: If $0 \leq k \leq n$, then

$$C(n, k) = C(n, n - k).$$

Proof:

$$C(n, k) = \frac{n!}{k!(n-k)!} = \frac{n!}{(n-k)!(n-(n-k))!} = C(n, n-k)$$

Q: Why should choosing k things out of n be the same as choosing $n - k$ things out of n?

A: There's a 1-1 correspondence. For every way of choosing k things out of n, look at the things not chosen: that's a way of choosing $n - k$ things out of n.

This is a better way of thinking about Theorem 1 than the combinatorial proof.

Theorem 2: If $0 < k < n$ then

$$\binom{n}{k} = \binom{n-1}{k} + \binom{n-1}{k-1}$$

Proof 1: (Combinatorial) Suppose we want to choose k objects out of $\{1, \ldots, n\}$. Either we choose the last one (n) or we don't.

1. How many ways are there of choosing k without choosing the last one? $C(n-1, k)$.
2. How many ways are there of choosing k including n? This means choosing $k-1$ out of $\{1, \ldots, n-1\}$: $C(n-1, k-1)$.

Proof 2: Algebraic ...

Note: If we define $C(n, k) = 0$ for $k > n$ and $k < 0$, Theorems 1 and 2 still hold.

Pascal's Triangle

Starting with $n = 0$, the nth row has $n + 1$ elements:

$$C(n, 0), \ldots, C(n, n)$$

Note how Pascal's Triangle illustrates Theorems 1 and 2.

Theorem 3: For all $n \geq 0$:

$$\Sigma_{k=0}^{n} \binom{n}{k} = 2^n$$

Proof 1: $\binom{n}{k}$ tells you all the way of choosing a subset of size k from a set of size n. This means that the LHS is *all* the ways of choosing a subset from a set of size n. The product rule says that this is 2^n.

Proof 2: By induction. Let $P(n)$ be the statement of the theorem.

Basis: $\Sigma_{k=0}^{0}\binom{0}{k} = \binom{0}{0} = 1 = 2^0$. Thus $P(0)$ is true.

Inductive step: How do we express $\Sigma_{k=0}^{n} C(n,k)$ in terms of $n-1$, so that we can apply the inductive hypothesis?

- Use Theorem 2!

More combinatorial identities

Theorem 4: For any nonnegative integer n

$$\Sigma_{k=0}^{n} k \binom{n}{k} = n2^{n-1}$$

Proof 1:

$$\begin{aligned}
& \Sigma_{k=0}^{n} k \tbinom{n}{k} \\
= \; & \Sigma_{k=1}^{n} k \tfrac{n!}{(n-k)!k!} \\
= \; & \Sigma_{k=1}^{n} \tfrac{n!}{(n-k)!(k-1)!} \\
= \; & n\Sigma_{k=1}^{n} \tfrac{(n-1)!}{(n-k)!(k-1)!} \\
= \; & n\Sigma_{j=0}^{n-1} \tfrac{(n-1)!}{(n-1-j)!j!} \quad \text{[Let } j = k-1\text{]} \\
= \; & n\Sigma_{j=0}^{n-1} \tbinom{n-1}{j} \\
= \; & n2^{n-1}
\end{aligned}$$

Proof 2: LHS tells you all the ways of picking a subset of k elements out of n (a subcommittee) and designating one of its members as special (subcomittee chairman).

What's another way of doing this? Pick the chairman first, and then the rest of the subcommittee!

Theorem 5:

$$(n-k)\binom{n}{k} = (k+1)\binom{n}{(k+1)} = n\binom{(n-1)}{k}$$

Theorem 6:

$$\begin{aligned} C(n,k)C(n-k,j) &= C(n,j)C(n-j,k) \\ &= C(n,k+j)C(k+j,j) \end{aligned}$$

Theorem 7: $P(n,k) = nP(n-1,k-1)$.

The Binomial Theorem

We want to compute $(x+y)^n$.
Some examples:

$$(x+y)^0 = 1$$
$$(x+y)^1 = x+y$$
$$(x+y)^2 = x^2+2xy+y^2$$
$$(x+y)^3 = x^3+3x^2y+3xy^2+y^3$$
$$(x+y)^4 = x^4+4x^3y+6x^2y^2+4xy^3+y^4$$

The pattern of the coefficients is just like that in the corresponding row of Pascal's triangle!

Binomial Theorem:

$$(x+y)^n = \Sigma_{k=0}^{n}\binom{n}{k}x^{n-k}y^k$$

Proof 1: By induction on n. $P(n)$ is the statement of the theorem.

Basis: $P(1)$ is obviously OK. (So is $P(0)$.)

Inductive step:

$$
\begin{aligned}
& (x+y)^{n+1} \\
= \ & (x+y)(x+y)^n \\
= \ & (x+y)\Sigma_{k=0}^{n}\binom{n}{k}x^{n-k}y^k \\
= \ & \Sigma_{k=0}^{n}\binom{n}{k}x^{n-k+1}y^k + \Sigma_{k=0}^{n}\binom{n}{k}x^{n-k}y^{k+1} \\
= \ & \ldots \qquad\qquad \text{[Lots of missing steps]} \\
= \ & y^{n+1} + \Sigma_{k=0}^{n}(\binom{n}{k} + \binom{n}{k-1})x^{n-k+1}y^k \\
= \ & y^{n+1} + \Sigma_{k=0}^{n}(\binom{n+1}{k}x^{n+1-k}y^k \\
= \ & \Sigma_{k=0}^{n+1}\binom{n+1}{k}x^{n+1-k}y^k
\end{aligned}
$$

Proof 2: What is the coefficient of the $x^{n-k}y^k$ term in $(x+y)^n$?

Using the Binomial Theorem

Q: What is $(x+2)^4$?

A:

$$\begin{aligned} & (x+2)^4 \\ = \; & x^4 + C(4,1)x^3(2) + C(4,2)x^2 2^2 + C(4,3)x 2^3 + 2^4 \\ = \; & x^4 + 8x^3 + 24x^2 + 32x + 16 \end{aligned}$$

Q: What is $(1.02)^7$ to 4 decimal places?

A:

$$\begin{aligned} & (1+.02)^7 \\ = \; & 1^7 + C(7,1)1^6(.02) + C(7,2)1^5(.0004) + C(7,3)(.000008) + \\ = \; & 1 + .14 + .0084 + .00028 + \cdots \\ \approx \; & 1.14868 \\ \approx \; & 1.1487 \end{aligned}$$

Note that we have to go to 5 decimal places to compute the answer to 4 decimal places.

In the book they talk about the *multinomial theorem.* That's for dealing with $(x + y + z)^n$.

They also talk about the *binomial series theorem.* That's for dealing with $(x+y)^\alpha$, when α is any *real* number (like 0.3).

You're not responsible for these results.