Example of Extended Euclidean Algorithm

Recall that $\gcd(84, 33) = \gcd(33, 18) = \gcd(18, 15) = \gcd(15, 3) = \gcd(3, 0) = 3$

We work backwards to write 3 as a linear combination of 84 and 33:

$$
\begin{aligned}
3 &= 18 - 15 \\
&\qquad \text{[Now 3 is a linear combination of 18 and 15]} \\
&= 18 - (33 - 18) \\
&= 2(18) - 33 \\
&\qquad \text{[Now 3 is a linear combination of 18 and 33]} \\
&= 2(84 - 2 \times 33)) - 33 \\
&= 2 \times 84 - 5 \times 33 \\
&\qquad \text{[Now 3 is a linear combination of 84 and 33]}
\end{aligned}
$$

Some Consequences

Corollary 2: If a and b are relatively prime, then there exist s and t such that $as + bt = 1$.

Corollary 3: If $\gcd(a, b) = 1$ and $a \mid bc$, then $a \mid c$.

Proof:

- Exist $s, t \in Z$ such that $sa + tb = 1$
- Multiply both sides by c: $sac + tbc = c$
- Since $a \mid bc$, $a \mid sac + tbc$, so $a \mid c$

Corollary 4: If p is prime and $p \mid \Pi_{i=1}^{n} a_i$, then $p \mid a_i$ for some $1 \leq i \leq n$.

Proof: By induction on n:

- If $n = 1$: trivial.

Suppose the result holds for n and $p \mid \Pi_{i=1}^{n+1} a_i$.

- note that $p \mid \Pi_{i=1}^{n+1} a_i = (\Pi_{i=1}^{n} a_i)a_{n+1}$.
- If $p \mid a_{n+1}$ we are done.
- If not, $\gcd(p, a_{n+1}) = 1$.
- By Corollary 3, $p \mid \Pi_{i=1}^{n} a_i$
- By the IH, $p \mid a_i$ for some $1 \leq i \leq n$.

The Fundamental Theorem of Arithmetic, II

Theorem 3: Every $n > 1$ can be represented uniquely as a product of primes, written in nondecreasing size.

Proof: Still need to prove uniqueness. We do it by strong induction.

- Base case: Obvious if $n = 2$.

Inductive step. Suppose OK for $n' < n$.

- Suppose that $n = \Pi_{i=1}^{s} p_i = \Pi_{j=1}^{r} q_j$.
- $p_1 \mid \Pi_{j=1}^{r} q_j$, so by Corollary 4, $p_1 \mid q_j$ for some j.
- But then $p_1 = q_j$, since both p_1 and q_j are prime.
- But then $n/p_1 = p_2 \cdots p_s = q_1 \cdots q_{j-1} q_{j+1} \cdots q_r$
- Result now follows from I.H.

Characterizing the GCD and LCM

Theorem 6: Suppose $a = \Pi_{i=1}^{n} p_i^{\alpha_i}$ and $b = \Pi_{i=1}^{n} p_i^{\beta_i}$, where p_i are primes and $\alpha_i, \beta_i \in N$.

- Some α_i's, β_i's could be 0.

Then

$$\begin{aligned} \gcd(a,b) &= \Pi_{i=1}^{n} p_i^{\min(\alpha_i,\beta_i)} \\ \mathrm{lcm}(a,b) &= \Pi_{i=1}^{n} p_i^{\max(\alpha_i,\beta_i)} \end{aligned}$$

Proof: For gcd, let $c = \Pi_{i=1}^{n} p_i^{\min(\alpha_i,\beta_i)}$.
Clearly $c \mid a$ and $c \mid b$.

- Thus, c is a common divisor, so $c \leq \gcd(a,b)$.

If $q^\gamma \mid \gcd(a,b)$,

- must have $q \in \{p_1, \ldots, p_n\}$
  - Otherwise $q \not| a$ so $q \not| \gcd(a,b)$ (likewise b)

  If $q = p_i$, $q^\gamma \mid \gcd(a,b)$, must have $\gamma \leq \min(\alpha_i, \beta_i)$
  - E.g., if $\gamma > \alpha_i$, then $p_i^\gamma \not| a$
- Thus, $c \geq \gcd(a,b)$.

Conclusion: $c = \gcd(a,b)$.

For lcm, let $d = \Pi_{i=1}^{n} p_i^{\max(\alpha_i,\beta_i)}$.

- Clearly $a \mid d$, $b \mid d$, so d is a common multiple.
- Thus, $d \geq \mathrm{lcm}(a, b)$.

Suppose $\mathrm{lcm}(a, b) = \Pi_{i=1}^{n} p_i^{\gamma_i}$.

- Must have $\alpha_i \leq \gamma_i$, since $p_i^{\alpha_i} \mid a$ and $a \mid \mathrm{lcm}(a, b)$.
- Similarly, must have $\beta_i \leq \gamma_i$.
- Thus, $\max(\alpha_i, \beta_i) \leq \gamma_i$.

Conclusion: $d = \mathrm{lcm}(a, b)$.

Example: $432 = 2^4 3^3$, and $95256 = 2^3 3^5 7^2$, so

- $\gcd(95256, 432) = 2^3 3^3 = 216$
- $\mathrm{lcm}(95256, 432) = 2^4 3^5 7^2 = 190512$.

Corollary 5: $ab = \gcd(a, b) \cdot \mathrm{lcm}(a, b)$

Proof:

$$\min(\alpha, \beta) + \max(\alpha, \beta) = \alpha + \beta.$$

Example: $4 \cdot 10 = 2 \cdot 20 = \gcd(4, 10) \cdot \mathrm{lcm}(4, 10)$.

Modular Arithmetic

Remember: $a \equiv b \pmod{m}$ means a and b have the same remainder when divided by m.

- Equivalently: $a \equiv b \pmod{m}$ iff $m \mid (a - b)$
- a is *congruent* to b mod m

Theorem 7: If $a_1 \equiv a_2 \pmod{m}$ and $b_1 \equiv b_2 \pmod{m}$, then

(a) $(a_1 + b_1) \equiv (a_2 + b_2) \pmod{m}$

(b) $a_1 b_1 \equiv a_2 b_2 \pmod{m}$

Proof: Suppose

- $a_1 = c_1 m + r$, $a_2 = c_2 m + r$
- $b_1 = d_1 m + r'$, $b_2 = d_2 m + r'$

So

- $a_1 + b_1 = (c_1 + d_1)m + (r + r')$
- $a_2 + b_2 = (c_2 + d_2)m + (r + r')$

$m \mid ((a_1 + b_1) - (a_2 + b_2) = ((c_1 + d_1) - (c_2 + d_2))m$

- Conclusion: $a_1 + b_1 \equiv a_2 + b_2 \pmod{m}$.

For multiplication:

- $a_1 b_1 = (c_1 d_1 m + r' c_1 + r d_1) m + r r'$
- $a_2 b_2 = (c_2 d_2 m + r' c_2 + r d_2) m + r r'$

$m \mid (a_1 b_1 - a_2 b_2)$

- Conclusion: $a_1 b_1 \equiv a_2 b_2 \pmod{m}$.

Bottom line: addition and multiplication carry over to the modular world.

Modular arithmetic has lots of applications.

- Here are four ...

Hashing

Problem: How can we efficiently store, retrieve, and delete records from a large database?

- For example, students records.

Assume, each record has a unique key

- E.g. student ID, Social Security #

Do we keep an array sorted by the key?

- Easy retrieval but difficult insertion and deletion.

How about a table with an entry for every possible key?

- Often infeasible, almost always wasteful.
- There are 10^{10} possible social security numbers.

Solution: store the records in an array of size N, where N is somewhat bigger than the expected number of records.

- Store record with id k in location $h(k)$
  - h is the *hash function*
  - Basic hash function: $h(k) := k \pmod{N}$.
- A collision occurs when $h(k_1) = h(k_2)$ and $k_1 \neq k_2$.
  - Choose N sufficiently large to minimize collisions
- Lots of techniques for dealing with collisions

Pseudorandom Sequences

For randomized algorithms we need a random number generator.

- Most languages provide you with a function "rand".
- There is nothing random about rand!
  - It creates an apparently random sequence deterministically
  - These are called *pseudorandom sequences*

A standard technique for creating psuedorandom sequences: the *linear congruential method*.

- Choose a modulus $m \in N^+$,
- a multiplier $a \in \{2, 3, \ldots, m-1\}$, and
- an increment $c \in Z_m = \{0, 1, \ldots, m-1\}$.
- Choose a seed $x_0 \in Z_m$
  - Typically the time on some internal clock is used
- Compute $x_{n+1} = ax_n + c \pmod{m}$.

Warning: a poorly implemented rand, such as in C, can wreak havoc on Monte Carlo simulations.

ISBN Numbers

Since 1968, most published books have been assigned a 10-digit ISBN numbers:

- identifies country of publication, publisher, and book itself
- The ISBN number for DAM3 is 1-56881-166-7

All the information is encoded in the first 9 digits

- The 10th digit is used as a parity check
- If the digits are $a_1, \ldots, a_{10}$, then we must have
$$a_1 + 2a_2 + \cdots + 9a_9 + 10a_{10} \equiv 0 \pmod{11}.$$
- For DAM3, get
$$\begin{aligned}&1 + 2 \times 5 + 3 \times 6 + 4 \times 8 + 5 \times 8 + 6 \times 1\\&+7 \times 1 + 8 \times 6 + 9 \times 6 + 10 \times 7 = 286 \equiv 0 \pmod{11}\end{aligned}$$
- This test always detects errors in single digits and transposition errors
  - Two arbitrary errors may cancel out

Similar parity checks are used in universal product codes (UPC codes/bar codes) that appear on almost all items

- The numbers are encoded by thicknesses of bars, to make them machine readable

Casting out 9s

Notice that a number is equivalent to the sum of its digits mod 9. This can be used as a way of checking your addition and of doing mindreading [come to class to hear more . . .]

Linear Congruences

The equation $ax = b$ for $a, b \in R$ is uniquely solvable if $a \neq 0$: $x = ba^{-1}$.

- Can we also (uniquely) solve $ax \equiv b \pmod{m}$?
- If x_0 is a solution, then so is $x_0 + km\ \forall k \in Z$
  - …since $km \equiv 0 \pmod{m}$.

So, uniqueness can only be mod m.

But even mod m, there can be more than one solution:

- Consider $2x \equiv 2 \pmod{4}$
- Clearly $x \equiv 1 \pmod{4}$ is one solution
- But so is $x \equiv 3 \pmod{4}$!

Theorem 8: If $\gcd(a, m) = 1$ then there is a unique solution (mod m) to $ax \equiv b \pmod{m}$.

Proof: Suppose $r, s \in Z$ both solve the equation:

- then $ar \equiv as \pmod{m}$, so $m \mid a(r - s)$
- Since $\gcd(a, m) = 1$, by Corollary 3, $m \mid (r - s)$
- But that means $r \equiv s \pmod{m}$

So if there's a solution at all, then it's unique mod m.

Solving Linear Congruences

But why is there a solution to $ax \equiv b \pmod{m}$?

Key idea: find $a^{-1} \bmod m$; then $x \equiv ba^{-1} \pmod{m}$

- By Corollary 2, since $\gcd(a, m) = 1$, there exist s, t such that
$$as + mt = 1$$
- So $as \equiv 1 \pmod{m}$
- That means $s \equiv a^{-1} \pmod{m}$
- $x \equiv bs \pmod{m}$

The Chinese Remainder Theorem

Suppose we want to solve a system of linear congruences:

Example: Find x such that

$$\begin{aligned} x &\equiv 2 \pmod 3 \\ x &\equiv 3 \pmod 5 \\ x &\equiv 2 \pmod 7 \end{aligned}$$

Can we solve for x? Is the answer unique?

Definition: $m_1, \ldots, m_n$ are *pairwise relatively prime* if each pair m_i, m_j is relatively prime.

Theorem 9 (Chinese Remainder Theorem): Let $m_1, \ldots, m_n \in N^+$ be pairwise relatively prime. The system

$$x \equiv a_i \pmod{m_i} \qquad i = 1, 2 \ldots n \tag{1}$$

has a unique solution modulo $M = \Pi_1^n m_i$.

- The best we can hope for is uniqueness modulo M:
  - If x is a solution then so is $x + kM$ for any $k \in Z$.

Proof: First I show that there is a solution; then I'll show it's unique.

CRT: Existence

Key idea for existence:
Suppose we can find $y_1, \ldots, y_n$ such that

$$\begin{aligned} y_i &\equiv a_i \pmod{m_i} \\ y_i &\equiv 0 \pmod{m_j} \quad \text{if } j \neq i. \end{aligned}$$

Now consider $y := \Sigma_{j=1}^n y_j$.

$$\Sigma_{j=1}^n y_j \equiv a_i \pmod{m_i}$$

- Since $y_i = a_i \bmod m_i$ and $y_j = 0 \bmod m_j$ if $j \neq i$.

So y is a solution!

- Now we need to find $y_1, \ldots, y_n$.
- Let $M_i = M/m_i = m_1 \times \cdots \times m_{i-1} \times m_{i+1} \times \cdots \times m_n$.
- $\gcd(M_i, m_i) = 1$, since m_j's pairwise relatively prime
  - No common prime factors among any of the m_j's

  Choose y_i' such that $(M_i)y_i' \equiv a_i \pmod{m_i}$
  - Can do that by Theorem 8, since $\gcd(M_i, m_i) = 1$.

  Let $y_i = y_i' M_i$.
  - y_i is a multiple of m_j if $j \neq i$, so $y_i \equiv 0 \pmod{m_j}$
  - $y_i = y_i' M_i \equiv a_i \pmod{m_i}$ by construction.

So $y_1 + \cdots + y_n$ is a solution to the system, mod M.

CRT: Example

Find x such that

$$x \equiv 2 \pmod{3}$$
$$x \equiv 3 \pmod{5}$$
$$x \equiv 2 \pmod{7}$$

Find y_1 such that $y_1 \equiv 2 \pmod{3}$, $y_1 \equiv 0 \pmod{5/7}$:

- y_1 has the form $y_1' \times 5 \times 7$
- $35y_1' \equiv 2 \pmod{3}$
- $y_1' = 1$, so $y_1 = 35$.

Find y_2 such that $y_2 \equiv 3 \pmod{5}$, $y_2 \equiv 0 \pmod{3/7}$:

- y_2 has the form $y_2' \times 3 \times 7$
- $21y_2' \equiv 3 \pmod{5}$
- $y_2' = 3$, so $y_2 = 63$.

Find y_3 such that $y_3 \equiv 2 \pmod{7}$, $y_3 \equiv 0 \pmod{3/5}$:

- y_3 has the form $y_3' \times 3 \times 5$
- $15y_3' \equiv 2 \pmod{7}$
- $y_3' = 2$, so $y_3 = 30$.

Solution is $x = y_1 + y_2 + y_3 = 35 + 63 + 30 = 128$

CRT: Uniqueness

What if x, y are both solutions to the equations?

- $x \equiv y \pmod{m_i} \Rightarrow m_i \mid (x - y)$, for $i = 1, \ldots, n$
- **Claim:** $M = m_1 \cdots m_n \mid (x - y)$
- so $x \equiv y \pmod{M}$

Theorem 10: If $m_1, \ldots, m_n$ are pairwise relatively prime and $m_i \mid b$ for $i = 1, \ldots, n$, then $m_1 \cdots m_n \mid b$.

Proof: By induction on n.

- For $n = 1$ the statement is trivial.

Suppose statement holds for $n = N$.

- Suppose $m_1, \ldots, m_{N+1}$ relatively prime, $m_i \mid b$ for $i = 1, \ldots, N + 1$.
- by IH, $m_1 \cdots m_N \mid b \Rightarrow b = m_1 \cdots m_N c$ for some c
- By assumption, $m_{N+1} \mid b$, so $m \mid (m_1 \cdots m_N)c$
- $\gcd(m_1 \cdots m_N, m_{N+1}) = 1$ (since m_i's pairwise relatively prime $\Rightarrow$ no common factors)
- by Corollary 3, $m_{N+1} \mid c$
- so $c = dm_{N+1}$, $b = m_1 \cdots m_N m_{N+1} d$
- so $m_1 \cdots m_{N+1} \mid b$.

An Application of CRT: Computer Arithmetic with Large Integers

Suppose we want to perform arithmetic operations (addition, multiplication) with extremely large integers

- too large to be represented easily in a computer

Idea:

- Step 1: Find suitable moduli $m_1, \ldots, m_n$ so that m_i's are relatively prime and $m_1 \cdots m_n$ is bigger than the answer.
- Step 2: Perform all the operations mod m_j, $j = 1, \ldots, n$.
  - This means we're working with much smaller numbers (no bigger than m_j)
  - The operations are much faster
  - Can do this in parallel
- Suppose the answer mod m_j is a_j:
  - Use CRT to find x such that $x \equiv a_j \pmod{m_j}$
  - The unique x such that $0 < x < m_1 \cdots m_n$ is the answer to the original problem.

Example: The following are pairwise relatively prime:

$$2^{35}-1,\ 2^{34}-1,\ 2^{33}-1,\ 2^{29}-1,\ 2^{23}-1$$

We can add and multiply positive integers up to

$$(2^{35}-1)(2^{34}-1)(2^{33}-1)(2^{29}-1)(2^{23}-1) > 2^{163}.$$

Fermat's Little Theorem

Theorem 11 (Fermat's Little Theorem):
(a) If p prime and $\gcd(p, a) = 1$, then $a^{p-1} \equiv 1 \pmod{p}$.

(b) For all $a \in Z$, $a^p \equiv a \pmod{p}$.

Proof. Let

$$A = \{1, 2, \ldots, p-1\}$$
$$B = \{1a \bmod p, 2a \bmod p, \ldots, (p-1)a \bmod p\}$$

Claim: $A = B$.

- $0 \notin B$, since $p \not| \, ja$, so $B \subset A$.
- If $i \neq j$, then $ia \bmod p \neq ja \bmod p$
  - since $p \not| \, (j-i)a$

Thus $|A| = p - 1$, so $A = B$.

Therefore,

$$\Pi_{i \in A}\, i \equiv \Pi_{i \in B}\, i \pmod{p}$$
$$\Rightarrow \; (p-1)! \equiv a(2a) \cdots (p-1)a = (p-1)!\, a^{p-1} \pmod{p}$$
$$\Rightarrow \; p \mid (a^{p-1} - 1)(p-1)!$$
$$\Rightarrow \; p \mid (a^{p-1} - 1) \quad [\text{since } \gcd(p, (p-1)!) = 1]$$
$$\Rightarrow \; a^{p-1} \equiv 1 \pmod{p}$$

It follows that $a^p \equiv a \pmod{p}$

- This is true even if $\gcd(p, a) \neq 1$; i.e., if $p \mid a$

Why is this being taught in a CS course?