CS280, Spring 2004: Prelim

The test is out of 50; the points for each question are marked. Don't forget to put your name and student number on each blue book that you use. You can answer the questions in any order, but mark your work clearly. Don't forget to show all your work. Give us a chance to give you partial credit!

Good luck.

1. [3 points] What is the transitive closure of the relation

$$\{(1,2),(2,3),(3,1),(3,4)\}?$$

2. [4 points] If f is a function from A to B, and S and T are subsets of B, prove that $f^{-1}(S \cap T) = f^{-1}(S) \cap f^{-1}(T)$.

3. [6 points] Suppose that R_1 and R_2 are both relations on N, the natural numbers. True or false:

   (a) if R_1 and R_2 are transitive relations, then so is $R_1 \cup R_2$.

   (b) if R_1 and R_2 are reflexive relations, then so is $R_1 \cup R_2$.

   In each case, if you think it's true, prove it. If not, give a counterexample.

4. [8 points] Suppose the sets $P_0, P_1, P_2, \ldots$ of bit strings (that is, strings of 0s and 1s) are defined inductively by taking $P_0 = \{\lambda\}$ (where λ denotes the empty string) and $P_{n+1} = P_n \cup \{x00, x01, x10, x11 : x \in P_n\}$. Let $P = \cup_{k=1}^{\infty} P_k$.

   Let Q be the smallest set such that

   (a) $\lambda \in Q$;

   (b) if $x \in Q$, then $x00, x01, x10, x11 \in Q$.

   Prove that $P = Q$.

5. [3 points] Canada has a two-dollar coin known colloquially as a "toonie". (The one-dollar coin, which has a picture of a loon on it, is called a "loonie".) What is wrong with the following argument, which purports to show that any debt of $n > 1$ Canadian dollars can be repaid (exactly) using only toonies?

We proceed by strong induction. Let $P(k)$ be the statement that a debt of k dollars can be repaid exactly using only toonies.

The base case is $k = 2$. Clearly a debt of \$2 Canadidan can be repaid with one toonie.

Assume that $P(k)$ is true for $k = 2, \ldots, n$. We now prove $P(n+1)$. By the induction hypothesis, a debt of $n-1$ dollars can be repaid exactly using toonies, by the induction hypothesis. Using one more toonie, the debt of $n+1$ dollars can be repaid.

6. [8 points] For $n \geq 0$, let $F_n = 2^{2^n} + 1$. (Those numbers F_n which are prime are called *Fermat primes*.)

   (a) [4 points] Prove by induction that $\prod_{r=0}^{n-1} F_r = F_n - 2$ for $n \geq 1$.

   (b) [4 points] Prove that $\gcd(F_m, F_n) = 1$ for all m, n with $m < n$. (Hint: use part (a)—which you can use even if you haven't proved it—and some standard facts about divisibility.)

7. [6 points]

   - [2 points] Find all solutions modulo 11 to the quadratic congruence: $x^2 \equiv 1 \pmod{11}$.
   - [4 points] Find all solutions modulo p to the quadratic congruence: $x^2 \equiv 1 \pmod{p}$ when p is prime. (It's not enough to list the solutions; you must *prove* that you have found them all.)

8. [6 points]

   (a) [2 points] Define gcd and lcm.

   (b) [4 points] Compute lcm(11413,12827).

9. [6 points] Suppose $A = \{a_1, \ldots, a_n\}$ and $B = \{0, 1\}$.

   (a) [3 points] Show that there are 2^n functions from A to B. (Hint: you don't need induction!)

   (b) [3 points] Show that there are $2^n - 2$ surjective functions from A to B.