Discrete Structures	**Problem Set 3**
CS 280 Spring 2005	**Due Wednesday, February 16, 2005**

(1) Consider the following proposed proof.

For all integers $n \geq 1$ we have that $f(n) = n(n+3)$ is odd.

Proposed proof. Let $P(n)$ be the statement that "$n(n+3)$ is odd". We prove that $P(n)$ implies $P(n+1)$ and hence $P(n)$ is true for all n.

To see that $P(n)$ implies $P(n+1)$ consider the expression $f(n+1)$. To use the induction hypothesis, we want to write $f(n+1)$ using $f(n)$. So note that $f(n+1) = (n+1)(n+4) = n(n+3) + 2n + 4 = f(n) + 2n + 4$. Now $2n + 4$ is even for all integers n. By the induction hypothesis $f(n)$ is odd, and hence $P(n+1)$ is the sum of an even and an odd number, and so it is also odd.

However, something must be wrong with this proof, as $P(n)$ is not true for all n. For example, if you take $n = 7$ then $f(n) = n(n+3) = 70$ which is even.

Explain where the mistake is.

(2) Consider the following proposed proof.

For all integers $n \geq 0$, and all numbers $a > 0$ we have that $a^n = 1$.

Proposed proof. We use strong induction on n, where $P(n)$ is the statement that "For all numbers $a > 0$ we have that $a^n = 1$".

Base case $a^0 = 1$ is true by definition.

Now consider the inductive step. We want to show that if $P(k)$ is true for all values $k \leq n$ then $P(n+1)$ is also true. So write

$$a^{n+1} = \frac{a^n \cdot a^n}{a^{n-1}} = \frac{1 \cdot 1}{1} = 1,$$

where the second equation is using the induction hypothesis $P(n)$ for the numerator and $P(n-1)$ for the denominator.

Explain what is wrong with the proof.

(3) Consider a sport in which two teams play, and they accumulate points in increments of 3 and 7 only. Show that for all natural numbers $n \geq 12$ it is possible to get n points.

For example, they can get 12 points (by scoring 3 repeatedly), 13 points (by scoring $7+3+3$), but 8 or 11 are not possible.

(4) Recall the definition of the Harmonic number $H(n) = 1 + \frac{1}{2} + \frac{1}{2} + \ldots + \frac{1}{n}$. In class we used induction to show that for all integers $k \geq 0$ we have that $H(2^k) \geq k/2$, which then implied that $H(n) \geq \frac{1}{2}\lfloor \log_2 n \rfloor$ for all $n \geq 1$.

Here we would like to have an (almost) matching upper bound. For this end we will need the upper integer part function. For an number α we use $\lceil \alpha \rceil$ to denote the smallest integer no smaller than α (or equivalently the smallest integer greater than or equal to α). So for example, $\lceil 8.213.. \rceil = 9$, while $\lceil 8 \rceil = 8$.

Use induction to show that our lower bound is tight up to a factor of 2, namely $H(2^k) \leq k+1$ for all $k \geq 0$, and show that this implies that $H(n) \leq \lceil \log_2 n \rceil + 1$ for all $n \geq 1$.

(5) Consider the ordering problem we discussed in class. So there are n Web pages, and we have pairwise comparisons indicating majority preferences between every pair of pages. Here we want to consider a way to select a best page. So obviously, a page X is "best" if the following is true.

$B(X)$= "*$\forall$ pages $Y \neq X$ the majority prefers X to Y.*"

As we have seen in class, there may not be a page X with this property. So here we consider a weaker property:

$B'(X)$= "*$\forall$ pages $Y \neq X$ (either the majority prefers X to Y) or (there is a page Z such that (the majority prefers X to Z) and (the majority prefers Z to Y)).*"

Show by induction on n that for all preferences, there is always a page X that satisfies $B'(X)$.

(6) Define a_n for each natural number n as follows. $a_0 = 1$, and $a_{n+1} = \sqrt{(1 + a_n)}$. For example, $a_1 = \sqrt{2}$, $a_2 = \sqrt{(1 + \sqrt{2})}$. We'll see in class that $a_1 = \sqrt{2}$ is not rational. Prove by using induction on n that for $n \geq 1$ a_n is not rational. (You do not have to show that $\sqrt{2}$ is not rational).

(7 optional) In understanding displays on the screen one often has to consider arrangements of lines, and the regions of the plane they define. Consider a plane divided by n lines. We will be wondering how many regions we get using n lines. For example, when $n = 1$, we get two regions on the two sides of the plane. When $n = 2$ we get 3 regions if the two lines are parallel and 4 regions if they are not. Suppose you have n lines, so that no two are parallel, and no 3 go through the same point. How many regions do such lines divide the plane to? So $r(1) = 2$, and $r(2) = 4$. Express this number as a closed form function $r(n)$ of the number of lines n. Use induction to prove that your answer is correct.

What makes this question hard is that I did not give you the formula for the number of regions. So your first problem is to figure out what it may be. Here is how you can do that. Try to think about the induction step in the proof, which will depend on how $r(n+1)$ differs from $r(n)$. Once you know this, use $r(1) = 2$ or $r(2) = 4$ to figure out what this induction step proves.