# CS 280, Spring 1998: Final Exam Solutions

1. An answer is just a sequence of 7 Ns and 4 Es, listed in any order. There are $\frac{11!}{7!4!} = \frac{11\times 10\times 9\times 8}{4\times 3\times 2} = 330$ such sequences, so that's how many paths there are.

2. Using the pigeonhole principle, taking the students to be the pigeons and the days of the week to be the holes. There are seven holes, since there are seven days of the week. We need eight pigeons (excuse me, students) to guarantee that at least two must go in the same hole.

3. Here's the algebraic argument:

$$\begin{aligned} \binom{2n}{2} &= \frac{2n!}{2n-1)!2!} \\ &= \frac{2n(2n-1)}{2} \\ &= n(2n-1) \\ &= n(n-1) + n^2 \\ &= 2\binom{n}{2} + n^2 \end{aligned}$$

   Here's the combinatorial argument:
   Suppose you want to choose 2 things out of $2n$. Divide the $2n$ things into two subsets of size $n$; call these subsets $A$ and $B$. To choose 2 things out of the whole set, you can either choose both from $A$, both from $B$, or one from $A$ and one from $B$. There are $\binom{n}{2}$ ways of choosing 2 things from $A$, another $\binom{n}{2}$ ways of choosing two things from $B$, and $n^2$ ways of choosing one thing from $A$ and one from $B$. This shows that

$$\binom{2n}{2} = 2\binom{n}{2} + n^2.$$

4. There are $C(10,k)$ bit strings with exactly $k$ 1s in them. Thus, there are $C(10,8) + C(10,9) + C(10,10) = 45 + 10 + 1 = 56$ bit strings with at least 8 1s in them. [You lost .5 if you didn't simplify this to 56.]

5. Let $P(n)$ be the statment of the problem, that is $P(n)$ is $\sum_{j=n}^{2n-1}(2j+1) = 3n^2$. (You lost .5 if you included "if $n \geq 1$" as part of $P(n)$.) We prove $P(n)$ for $n \geq 1$ by induction on $n$.

   *Basis:* P(1) says $\sum_{j=1}^{1} 2j + 1 = 3$. This is clearly true.

Assume $P(n)$. We prove $P(n+1)$. Note that $P(n+1)$ says $\sum_{j=n+1}^{2n+1}(2j+1) = 3(n+1)^2$.

$$\begin{array}{rl} & \sum_{j=n+1}^{2n+1}(2j+1) \\ = & \sum_{j=n}^{2n-1}(2j+1) - (2n+1) + (4n+1) + (4n+3) \\ = & 3n^2 + 6n + 3 \qquad \text{[Using } P(n)\text{]} \\ = & 3(n^2 + 2n + 1) \\ = & 3(n+1)^2 \end{array}$$

Thus, $P(n) \Rightarrow P(n+1)$, and we're done.

6. Using the rules for converting to DNF discussed in class, we have:

$$\begin{array}{rl} & (x+y) \cdot \overline{y \cdot \overline{z}} \\ = & (x+y) \cdot (\overline{y} + \overline{\overline{z}}) \\ = & (x+y) \cdot (\overline{y} + z) \\ = & x \cdot \overline{y} + x \cdot z + y \cdot \overline{y} + y \cdot z \\ = & x \cdot \overline{y} + x \cdot z + 0 + y \cdot z \\ = & x \cdot \overline{y} + x \cdot z + y \cdot z \end{array}$$

You could have stopped at $x \cdot \overline{y} + x \cdot z + y \cdot z$ for full credit, or kept going as in the text as follows:

$$\begin{array}{rl} & x \cdot \overline{y} + x \cdot z + y \cdot z \\ = & x \cdot \overline{y} \cdot (z + \overline{z}) + x \cdot z \cdot (y + \overline{y}) + y \cdot z \cdot (x + \overline{x}) \\ = & x \cdot \overline{y} \cdot z + x \cdot \overline{y} \cdot \overline{z} + x \cdot y \cdot z + x \cdot \overline{y} \cdot z + x \cdot y \cdot z + \overline{x} \cdot y \cdot z \\ = & x \cdot \overline{y} \cdot z + x \cdot \overline{y} \cdot \overline{z} + x \cdot y \cdot z + \overline{x} \cdot y \cdot z \end{array}$$

If you had $x \cdot \overline{y} + x \cdot z + y \cdot z$ but made a mistake in the computations after that, you lost .5 or 1, depending on the mistake. (Sometimes it's good to know when to quit.)

7. (a) (i) Every number is less than or equal to some other number.
   (ii) There is a number that is less than or equal to every number (i.e., there is a smallest number).
   (iii) There is a number such that every number is less than or equal to it (i.e., there is a largest number).

   (b) If we take as the domain the natural numbers, (ii) is true but (iii) is false.

   (c) The negative integers is a domain for which (iii) is true but (ii) is false.

   [If your answers for (b) and (c) were consistent with your answers in (a), you received full credit for (b) and (c), even if your answers to (a) were wrong.]

8. The domain $C$ consist of all the cars in this centrally isolated town. Let $SUV$ be the subset of $C$ that are SUVs, and let $ST$ be the subset of cars in $C$ that are owned by students. If we identify probability with statistics, then the problem tells us that $\Pr(ST) = \frac{1}{3}$ (so $\Pr(\overline{ST} = \frac{2}{3})$, $\Pr(SUV|ST) = .35$, and $\Pr(SUV|\overline{ST}) = .2$. The problem asks you to compute $\Pr(ST|SUV)$. [You got 1 point for identifying the appropriate domain ($C$), and another one point if you identified the appropriate subsets $SUV$ and $ST$. You got .5 if you identified one of $SUV$ and $ST$ but not the other. You lost a point if you didn't identify the domain, or if your "appropriate subsets" weren't subsets of the domain you did identify. You got yet another point for observing that the problem tells you that $\Pr(ST) = \frac{1}{3}$, $\Pr(SUV|ST) = .35$, $\Pr(SUV|\overline{ST}) = .2$, and .5 if you got 2 out of 3 of these. Finally, you got .5 for realizing that you needed to compute $\Pr(ST|SUV)$. Thus, you could get 3.5/5 even if you did not know how to compute the answer.) To compute $\Pr(ST|SUV)$ we use Bayes' Rule:

$$\begin{aligned}\Pr(ST|SUV) &= \frac{\Pr(SUV|ST)\times\Pr(ST)}{\Pr(SUV|ST)\times\Pr(ST)+\Pr(SUV|\overline{ST})\times\Pr(\overline{ST})} \\ &= \frac{.35\times\frac{1}{3}}{.35\times\frac{1}{3}+.2\times\frac{2}{3}} \\ &= \frac{.35}{.75} \\ &= \frac{7}{15}\end{aligned}$$

9. If $P$ and $Q$ are false, then so are both $(P \vee Q)$ and $(P \wedge Q)$. Since $A \Rightarrow B$ is vacuously true if $A$ is false, and the $A$ in this case is $P \vee Q$, which is false, the truth value of $(P \vee Q) \Rightarrow (P \wedge Q)$ is true.

10. (a) We need to show that $x \in A - (B \cap C)$ iff $x \in (A - B) \cup (A - C)$.

$$\begin{aligned} & x \in A - (B \cap C) \\ \text{iff} \quad & x \in A \text{ and } x \notin B \cap C \\ \text{iff} \quad & x \in A \text{ and } (x \notin B \text{ or } x \notin C) \\ \text{iff} \quad & (x \in A \text{ and } x \notin B) \text{ or } (x \in A \text{ and } x \notin C) \\ \text{iff} \quad & (x \in A - B) \text{ and } (x \in A - C) \\ \text{iff} \quad & x \in ((A - B) \cup (A - C))\end{aligned}$$

(b) Here is the truth table for $P \wedge \neg(Q \wedge R)$:

| $P$ | $Q$ | $R$ | $Q \wedge R$ | $\neg(Q \wedge R)$ | $P \wedge \neg(Q \wedge R)$ |
|---|---|---|---|---|---|
| T | T | T | T | F | F |
| T | T | F | F | T | T |
| T | F | T | F | T | T |
| T | F | F | F | T | T |
| F | T | T | T | F | F |
| F | T | F | F | T | F |
| F | F | T | F | T | F |
| F | F | F | F | T | F |

Here is the truth table for $(P \wedge \neg Q) \vee (P \wedge \neg R)$:

| $P$ | $Q$ | $R$ | $\neg Q$ | $\neg R$ | $P \wedge \neg Q$ | $P \wedge \neg R$ | $(P \wedge \neg Q) \vee (P \wedge \neg R)$ |
|---|---|---|---|---|---|---|---|
| T | T | T | F | F | F | F | F |
| T | T | F | F | T | F | T | T |
| T | F | T | T | F | T | F | T |
| T | F | F | T | T | T | T | T |
| F | T | T | F | F | F | F | F |
| F | T | F | F | T | F | F | F |
| F | F | T | T | F | F | F | F |
| F | F | F | T | T | F | F | F |

From the truth tables, it is clear that every truth assignment to $P$, $Q$, and $R$ makes $P \wedge \neg(Q \wedge R)$ true iff it makes $(P \wedge \neg Q) \vee (P \wedge \neg R)$ true. (The last column in both tables is the same.) Thus, the formulas are equivalent.

(c) Let $A$ (resp., $B$, $C$) be the set of truth assignments to $P$, $Q$, and $R$ that make $P$ (resp., $Q$, $R$) true. Then $A - (B \cap C)$ is the set of truth assginments that make $P \wedge \neg(Q \vee R)$ true and $(A - B) \cup (A - C)$ is the set of truth assignments that make $(P \wedge \neg Q) \vee (P \wedge \neg R)$ true. Thus, the fact that $A - (B \cap C) = (A - B) \cup (A - C)$ implies that the same set of truth assignments make both formulas true, i.e., they are equivalent. That is, part (b) follows from part (a), using this interpretation of $A$, $B$, and $C$.

11. (a) A reasonable sample space is $\{HHH, HHT, HTH, HTT, TH, TT\}$. Note that if the first element of the sequence is $H$, then the sequence has length 3, while if it is $T$, the sequence has length 2.

(b) Here is the tree:

(c) Let $T_1$ be the event that the coin lands tails on the first toss. This is the event $\{TH, TT\}$. (Remember, an event is a subset of the sample space.)

Let $T_2$ be the event that the coin lands tails on the second toss. This is the event $\{HTH, HTT, TT\}$.

$T_1 \cap T_2 = \{TT\}$ is the event that the coin lands heads on both tosses.

$\Pr(T_1) = 1/2$ (since the first coin tossed is fair).

$\Pr(T_2) = \Pr(\{HTH, HTT, TT\}) = \frac{1}{2} \times \frac{1}{3} \times \frac{2}{3} + \frac{1}{2} \times \frac{1}{3} \times \frac{1}{3} + \frac{1}{2} \times \frac{2}{3} = \frac{1}{2}$.

$\Pr(T_1 \cap T_2) = \Pr(\{TT\}) = 1/3$

Since $\Pr(T_1 \cap T_2) \neq \Pr(T_1) \times \Pr(T_2)$, these events are *not* independent.

[It was *not* enough in this problem to observe that the probability of tails on the second toss depended on how the first coin landed. You had to relate this to the definition of independence. People typically got 1.5/3 for observing that the probability of tails depended on the outcome of the first toss.]

(d) "Number of tails" can be viewed as a random variable since it can be viewed as a function from the sample space to the real numbers (actually, to $\{0, 1, 2\}$, since it's impossible to get three tails).

(e) The expected number of tails is, by definition: $\sum_k k \times \Pr(k \text{ tails}) = \Pr(1 \text{ tail}) + 2 \times \Pr(2 \text{ tails})$

$$\Pr(1 \text{ tail}) = \Pr(\{HTH, HHT, TH\}) = \frac{1}{2} \times \frac{1}{3} \times \frac{2}{3} + \frac{1}{2} \times \frac{2}{3} \times \frac{1}{3} + \frac{1}{2} \times \frac{1}{3} = \frac{7}{18}$$
$$\Pr(2 \text{ tails}) = \Pr(\{HTT, TT\}) = \frac{1}{18} + \frac{1}{3} = \frac{7}{18}$$

Thus, the expected number of tails is $\frac{7}{18} + 2 \times \frac{7}{18} = \frac{21}{18} = \frac{7}{6}$.

12. (a) $S \times T = \{(s, t) | s \in S, t \in T\}$.

(b) $S \times T \neq T \times S$. For example, let $S = \{1\}$ and $T = \{2\}$. Then $S \times T = \{(1, 2)\}$ and $T \times S = \{(2, 1)\}$. These are different sets (since $(1, 2) \neq (2, 1)$). [You got .5 if you said that $S \times T \neq T \times S$ and did not give a counterexample, or gave a counterexample we couldn't make sense of.]

(c) A relation on $S \times T$ is a subset of $S \times T$.

13. (a) The graphs are isomorphic. Consider the following bijection $f$ between the vertices in $G_1$ and the vertices in $G_2$:

- $f(a) = z$ (both of these vertices have degree 4)
- $f(b) = x$
- $f(c) = y$
- $f(d) = w$
- $f(e) = v$

(This mapping is not unique; we could have had $f(b) = w$ and $f(d) = x$ or $f(b) = y$ and $f(d) = v$, for example. What matters is that $b$ and $d$ map to diagonally opposite points in the rectangle and $c$ and $e$ map to the other two diagonally opposite points.) It is now easy to check that there is an edge between vertices $i$ and $j$ in graph $G_1$ iff there is an edge between $f(i)$ and $f(j)$ in $G_2$.

(b) The degree of vertex $b$ is 3.

14. The graph has an Eulerian circuit, since every node has even degree (degree 4, in fact). One Eulerian circuit in the graph is $a - b - d - c - a - e - c - e - d - a$.