Definitions

Suppose the group G acts on the set A. Then

(i) for $a \in A$, the orbit of a is the set $\mathcal{O}(a) := \{g \circ a \mid g \in G\}$, i.e., the set of all points of A reachable by a via G's action.

(ii) for $a \in A$, the stabilizer of a is the group $G_a := \{g \in G \mid g \circ a = a\}$, i.e., the subgroup of those elements of G which 'fix' a.

(iii) for $g \in G$, the fixed-point set of g is the set of all points in A 'fixed' by g, notated $\mathrm{Fix}(g) := \{a \in A \mid g \circ a = a\}$.

Remarks

Notice that $x \in \mathrm{Fix}(g)$ iff $g \in G_x$. Notice also that we can define an equivalence relation $a \sim b$ iff $b \in \mathcal{O}(a)$, so then the orbit of a is the equivalence class of a, hence A can be written as the disjoint union of orbits.

Lemma

For G acting on A we have $|\mathcal{O}(a)| = |G : G_a| \ \forall a \in A$.

Proof

Define $f : \mathcal{O}(a) \longrightarrow G/G_a$ by $f(g \circ a) := gG_a$.

If $g \circ a = g' \circ a$, then $(g^{-1}g') \circ a = a \Rightarrow g^{-1}g' \in G_a \Rightarrow gG_a = g'G_a$, so f is well-defined.

It's easy to check that f is also a bijection. //

By Lagrange's Theorem, we can see that $|G| = |\mathcal{O}(a)||G_a| \ \forall a \in A$. If there's only one orbit, so $A = \mathcal{O}(a) \ \forall a \in A$, then we say that G acts transitively on A. Further connecting orbits and stabilizers, we have...

Lemma

$\mathcal{O}(a) = \mathcal{O}(b) \Rightarrow a = g \circ b$, some $g \in G \Rightarrow G_b = gG_ag^{-1}$.

Proof

Exer. – show $G_b \subseteq gG_ag^{-1}$ and $gG_ag^{-1} \subseteq G_b$.