More Counting Stuff

As we've seen, in order to compute probabilities we often need to count sizes of sets. We can facilitate this by chopping the set into blocks, and then moving these blocks around using group actions, so we'll start with a review of the relevant theory.

Definition

Let G be a group and A be a set, then a group action of G on A is a function $\varphi : G \times A \longrightarrow A$, denoted $\varphi(g,a) := g \cdot a$ such that

(i) $g_1 \cdot (g_2 \cdot a) = (g_1 g_2) \cdot a \quad \forall g_1, g_2 \in G \ \forall a \in A$

(ii) $1_G \cdot a = a \quad \forall a \in A$.

Sometimes we'll choose $A = G$ and so let G 'act' on itself in some defined way.

Examples

- Let $A = \{1, 2, \ldots, n\}$ and $G = S_n$, the group of all permutations of n objects. Then we could define $g \cdot a$ to be the result of applying the permutation g to a.
- Let A be a cube in $\mathbb{R}^3$ and G be the group of all rotations of A which leave A occupying the same cube-sized space in $\mathbb{R}^3$ — the "group of symmetries" of A which preserve orientation (i.e., excluding reflections). Then define $g \cdot a$ as applying the rotation g to that point $a \in A$.
- Let G be any group and $A = G$ as a set. Then we could define $g \cdot a := ga$ (the element of A obtained by multiplying g and a in that order). Alternatively, we could choose to define $g \cdot a := gag^{-1}$ (the conjugation of a by g).