HOMEWORK 8 SOLUTIONS

PART A

1.(a) $a_n = a_{n-1} + 6\, a_{n-2}$, $a_0 = 3$, $a_1 = 6$

The characteristic equation of the recurrence relation is $r^2 - r - 6 = 0$
Its roots are r= 3 and r= -2. Hence the sequence $\{a_n\}$ is a solution to the recurrence relation if and only if

$$a_n = \alpha_1 3^n + \alpha_2 (-2)^n$$

for some constant α_1 and α_2.
From the initial condition, it follows that

$$a_0 = 3 = \alpha_1 + \alpha_2$$
$$a_1 = 6 = 3\alpha_1 - 2\alpha_2$$

Solving the equations, we get $\alpha_1 = 2.4$, $\alpha_2 = 0.6$
Hence the solution is the sequence $\{a_n\}$ with

$$a_n = 2.4.(3^n) + 0.6.(-2)^n$$

(b) $a_n = 7\, a_{n-1} - 10\, a_{n-2}$, $a_0 = 2$, $a_1 = 1$

The characteristic equation of the recurrence relation is $r^2 - 7r + 10 = 0$
Its roots are r= 2 and r= 5. Hence the sequence $\{a_n\}$ is a solution to the recurrence relation if and only if

$$a_n = \alpha_1 2^n + \alpha_2 5^n$$

for some constant α_1 and α_2.
From the initial condition, it follows that

$$a_0 = 2 = \alpha_1 + \alpha_2$$
$$a_1 = 1 = 2\alpha_1 + 5\alpha_2$$

Solving the equations, we get $\alpha_1 = 3$, $\alpha_2 = -1$
Hence the solution is the sequence $\{a_n\}$ with

$$a_n = 3.2^n - 5^n$$

(c) $a_n = 6\, a_{n-1} - 8\, a_{n-2}$, $a_0 = 4$, $a_1 = 10$

The characteristic equation of the recurrence relation is $r^2 - 6r + 8 = 0$
Its roots are r= 2 and r= 4. Hence the sequence $\{a_n\}$ is a solution to the recurrence relation if and only if

$$a_n = \alpha_1 2^n + \alpha_2 4^n$$

for some constant α_1 and α_2.
From the initial condition, it follows that

$$a_0 = 4 = \alpha_1 + \alpha_2$$
$$a_1 = 10 = 2\alpha_1 + 4\alpha_2$$

Solving the equations, we get $\alpha_1 = 3$, $\alpha_2 = 1$
Hence the solution is the sequence $\{a_n\}$ with

$$a_n = 3.2^n + 4^n$$

(d) $a_n = 2 a_{n-1} - a_{n-2}$, $a_0 = 4$, $a_1 = 1$

The characteristic equation of the recurrence relation is $r^2 -2r +1 = 0$
It only has one root, which is r= 1. Hence the sequence $\{a_n\}$ is a solution to the recurrence relation if and only if

$$a_n = \alpha_1 1^n + \alpha_2 (n)(1^n) = \alpha_1 + \alpha_2 n$$

for some constant α_1 and α_2.
From the initial condition, it follows that

$$a_0 = 4 = \alpha_1 + \alpha_2 (0)$$
$$a_1 = 1 = \alpha_1 + \alpha_2 (1)$$

Solving the equations, we get $\alpha_1 = 4$, $\alpha_2 = -3$
Hence the solution is the sequence $\{a_n\}$ with

$$a_n = 4 - 3n$$

2. (a) $a_n = 2 a_{n-1} + 2n^2$

The associated homogeneous equation is $a_n = 2 a_{n-1}$.
The characteristic equation is $r -2 = 0$. So r= 2. And its solution is

$a_n^{(h)} = \alpha 2^n$, where α is a constant.

$F(n) = 2n^2$, so there is a particular solution $a_n^{(p)}$ of the form $p_2 n^2 + p_1 n + p_0$
Substituting this term to the recurrence equation:

$$p_2 n^2 + p_1 n + p_0 = 2 (p_2 (n-1)^2 + p_1 (n-1) + p_0) + 2n^2$$
$$\iff p_2 n^2 + p_1 n + p_0 = 2p_2 n^2 - 4p_2 n + 2p_2 + 2p_1 n - 2 p_1 + 2p_0 + 2n^2$$

Simplifying, we'll get

$$(p_2 +2) n^2 + (p_1 - 4p_2) n + (2p_2 - 2p_1 + p_0) = 0$$

From the equation, we know that a particular solution will be when

$$p_2 + 2 = 0$$
$$p_1 - 4p_2 = 0$$
$$2p_2 - 2p_1 + p_0 = 0$$

, which means that $p_2 = -2$, $p_1 = -8$, $p_0 = -12$
Hence a particular solution is

$$a_n^{(p)} = -2 n^2 - 8 n - 12$$

So all solutions of the original recurrence relation are given by

$a_n = a_n^{(h)} + a_n^{(p)} = \alpha 2^n - 2 n^2 - 8 n - 12$, where α is a constant

(b) Find the solution of the equation when $a_1 = 4$

$$a_1 = 4 = \alpha (2) - 2 - 8 - 12$$

so $\alpha = 13$

The solution is $a_n = 13(2^n) - 2 n^2 - 8 n - 12$

PART B

3. $a_n = 7\, a_{n-1} - 16\, a_{n-2} + 12\, a_{n-3} + n\, 4^n$, $a_0 = -2$, $a_1 = 0$, $a_2 = 5$

The associated homogeneous equation is $a_n = 7\, a_{n-1} - 16\, a_{n-2} + 12\, a_{n-3}$.
The characteristic equation is $r^3 - 7r^2 + 16r - 12 = 0$.
So its roots are 2, 2, and 3. And its solution is

$a_n^{(h)} = (\alpha_0 + \alpha_1 n)2^n + \alpha_2 3^n$, where α_0, α_1, α_2 are constant.

$F(n) = n\, 4^n$,so there is a particular solution $a_n^{(p)}$ of the form $(p_1 n + p_0)\, 4^n$
Substituting this term to the recurrence equation:

$$(p_1 n + p_0)\, 4^n = 7(p_1(n-1) + p_0)\, 4^{n-1} - 16\, (p_1(n-2) + p_0)\, 4^{n-2} + 12(p_1(n-3) + p_0)\, 4^{n-3} + n\, 4^n$$

Dividing both sides with 4^{n-3} and simplifying, we'll get

$n\,(64 - 4p_1) - 4(\,5p_1 + p_0) = 0$

From the equation, we know that a particular solution will be when

$64 - 4p_1 = 0$
$5p_1 + p_0 = 0$

, which means that $p_1 = 16$, $p_0 = -80$
Hence a particular solution is

$a_n^{(p)} = (16\, n - 80)\, 4^n$

So all solutions of the original recurrence relation are given by

$a_n = a_n^{(h)} + a_n^{(p)} = (\alpha_0 + \alpha_1 n)2^n + \alpha_2 3^n + (16\, n - 80)\, 4^n$

where α_0, α_1, α_2 are constant

$a_0 = -2 = \alpha_0 + \alpha_2 - 80$ <=> $78 = \alpha_0 + \alpha_2$ **-(1)**
$a_1 = 0 = 2(\alpha_0 + \alpha_1) + 3\alpha_2 + (-64)\, 4$ <=> $256 = 2\alpha_0 + 2\alpha_1 + 3\alpha_2$ **-(2)**
$a_2 = 5 = 4(\alpha_0 + 2\alpha_1) + 9\alpha_2 + (-48)\, 16$ <=> $773 = 4\alpha_0 + 8\alpha_1 + 9\alpha_2$ **-(3)**

Eliminating α1 in equation **(2)** and **(3)**, we will get

$251 = 4\alpha_0 + 3\alpha_2$ **-(4)**

Solving **(1)** and **(4)**, we will get $\alpha_0 = 17$, $\alpha_1 = 19.5$, $\alpha_2 = 61$

So the solution is

$(17 + 19.5n).2^n + 61(3^n) + (16\, n - 80)\, 4^n$

4. (a) At the time n, we triple the no. of bacterias at time n-1, and also perish the 'too old' ones. However, we can't subtract a_{n-2} because this includes all the bacterias at time n-2. Instead we have to subtract the no. of NEW bacterias born at time n-2, or $2a_{n-3}$.

$a_n = 3\, a_{n-1} - 2a_{n-3}$
$= 2\, a_{n-1} + a_{n-1} - 2a_{n-3}$ ($2a_{n-3}$ is all the newly born bacteria at time n-2, so $a_{n-1} - 2a_{n-3}$ is all the newly born bacteria at time n-1, which is equal to $2a_{n-2}$)

$= 2 a_{n-1} + 2a_{n-2}$

(b) The characteristic equation of the recurrence relation is $r^2 -2r -2 = 0$
Solving r will show that the roots are $1\pm\sqrt{3}$ ($r1=2.732$ and $r_2= -0.732$)
Hence the solution to the recurrence relation is

$$a_n = \alpha_1 (1+\sqrt{3})^n + \alpha_2(1-\sqrt{3})^n$$

Since we know that $a_0 = 100$
We can then deduce that in the next hour, 200 new bacteria will be formed and no bacteria would have died yet, so the total no of bacteria $a_1 = 300$
$a_0= 100$
$a_1= 100$(existing) + 2 * 100(new) = 300
$a_2= 300$(existing) + 2 * 300(new) - 100(just-became-2-hr-old ones) = 800
$a_3= 800$(existing) + 2 * 800(new) - 200(just-became-2-hr-old ones) = 2200
$a_4= 2200$(existing)+ 2* 2200(new) - 600(just-became-2-hr-old ones)= 6000

$$a_0 = 100 = \alpha_1 + \alpha_2$$
$$a_1 = 300 = = \alpha_1 (1+\sqrt{3}) + \alpha_2(1-\sqrt{3})$$

Solving the equations, we will get $\alpha_1= 107.735$, $\alpha_2= -7.735$

So the solution for the equation is

$$a_n =107.735 (2.732^n) - 7.735(-0.732)^n$$

(c) The term $- 7.735(-0.732)^n)$ is insignificant compared to $107.735 (2.732^n)$ since it's very small as n gets larger. Thus we can ignore the term $-7.735(-0.732)^n$ in the equation to get an estimate no of hours

$107.735 (2.732^n) - 7.735(-0.732)^n > 1000,000$
Consider $107.735 (2.732^n) > 1000,000$
$2.732^n > 9,282$
$n \text{ Log } 2.732 > \text{Log } 9,282$
$n > 9.09$

However, this number is only an estimate, since previously we ignore the term $-17(0.382^{n)}$ Thus, we should check a_{10}, and a_9 or a_{11}
Checking $a_{10} =107.735 (2.732^{10}) - 7.735(-0.732)^{10} = 2,495,999$
$a_9 =107.735 (2.732^9) - 7.735(-0.732)^9 = 913,599$
So after 10 hours, the colony will contain more than 1 million bacteria

PART C

5. The no of elements in the union of the 7 sets =

$| S_1 | + | S_2| + \ldots + | S_7|$
$- (|S_1 \cap S_2| + |S_1 \cap S_3| + |S_1 \cap S_4| + \ldots + |S_6 \cap S_7|)$
$+ (|S_1 \cap S_2 \cap S_3| + |S_1 \cap S_3 \cap S_4| + \ldots + |S_1 \cap S_6 \cap S_7|)$
$- (|S_1 \cap S_2 \cap S_3 \cap S_4| + \ldots + |S_1 \cap S_3 \cap S_4 \cap S_5|)$
$+ (|S_1 \cap S_2 \cap S_3 \cap S_4 \cap S_5|) + \ldots + (|S_3 \cap S_4 \cap S_5 \cap S_6 \cap S_7|)$

The no of sets = 7
The no of ways of intersection of 2 sets = C(7,2)
The no of ways of intersection of 3 sets = C(7,3)
The no of ways of intersection of 4 sets = C(7,4)
The no of ways of intersection of 5 sets = C(7,5)
So the no of terms needed to express the no of elements

= 7 + C(7,2) + C(7,3) + C(7,4) + C(7,5)
= 119

6. Consider the set of strings using letters of the alphabet. For each of the relations below, determine if the relation is reflexive, irreflexive (defined on page 382), symmetric, antisymmetric, and/or transitive.

(a) {(a, b) | a and b are the same length}

It's **reflexive** since $\forall a \in A$, $(a, a) \in R$ (same string will always have the same length). It is **not irreflexive** since $\forall a \in A$, $(a, a) \in R$
It's **symmetric** since $\forall a,b \in A$,if $(a, b) \in R$ then $(b, a) \in R$ (If a has the same length as b, then b must have the same length as a)
It's **not antisymmetric** since if $(a, b) \in R$ and $(b, a) \in R$, it doesn't imply that a = b (Counter example : a= 'run', b= 'now' but $a \neq b$)
It is **transitive,** since if $(a, b) \in R$ and $(b, c) \in R$ then $(a, c) \in R$ $\forall a,b,c \in A$
(if a has the same length as b, and b has the same length as c, a and c must have same length too)

(b) {(a, b) | a and b both contain a vowel}

It's **not reflexive** since $\forall a \in A$, it's not always the case that $(a, a) \in R$ (Counter example: 'x'$\in$ A, but ('x', 'x') $\notin$ R). And it is **not irreflexive** since $\forall a \in A$, it's not always the case that $(a, a) \notin R$.(Counter example: 'e'$\in$ A, but ('e', 'e') $\in$ R).
It's **symmetric** since $\forall a,b \in A$,if $(a, b) \in R$ then $(b, a) \in R$ (If a and b both contain a vowel, then b and a both must contain a vowel too)
It's **not antisymmetric** since if $(a, b) \in R$ and $(b, a) \in R$, it doesn't imply that a = b (Counter example: a= 'the', b= 'end' but $a \neq b$)
It is **transitive**, since if $(a, b) \in R$ and $(b, c) \in R$ then $(a, c) \in R$ $\forall a, b, c \in A$
(if a and b both contains a vowel, and b and c both contains a vowel too, then a and c both must contain a vowel too)

(c) {(a, b) | a and b begin with different letters}

It's **not reflexive** since $\forall a \in A$, $(a, a) \notin R$ (same string will always contain the same letters, so it will never begin with different letters). It's **irreflexive** for the same reason.

It’s **symmetric** since $\forall a,b \in A$,if $(a, b) \in R$ then $(b, a) \in R$ (If a and b begin with different letters, then b and a both begin with different letters too)
It’s **not antisymmetric** since if $(a, b) \in R$ and $(b, a) \in R$, it doesn’t imply that a = b (Counter example : a= ‘hard’, b= ‘work’ but $a \neq b$)
It is **not transitive**, since if $(a, b) \in R$ and $(b, c) \in R$ then it is not always the case that $(a, c) \in R$ $\forall a,b,c \in A$ (Counter example: a=’time’, b=’mine’, c=’to’)

(d) {(a, b) | there is a letter in a that is not in b}

It’s **not reflexive** since $\forall a \in A, (a, a) \notin R$ (same string will not have a letter in it that is not in it). It’s **irreflexive** since $\forall a \in A, (a, a) \notin R$.
It’s **not symmetric** since $\forall a,b \in A$,if $(a, b) \in R$ then it’s not always the case that $(b, a) \in R$ (Counter example: a=’bend’, b=’end’)
It’s **not antisymmetric** since if $(a, b) \in R$ and $(b, a) \in R$, it doesn’t imply that a = b (Counter example : a= ‘arm’, b= ‘man’ but $a \neq b$)
It is **not transitive**, since if $(a, b) \in R$ and $(b, c) \in R$ then it is not always the case that $(a, c) \in R$ $\forall a,b,c \in A$ (Counter example: a=’end’, b=’door’, c=’bend’)