1. Reading: K. Rosen *Discrete Mathematics and Its Applications*, 6.1, 6.2.

2. The main message of this lecture:

As practically everything in Math and Computer Science the notion of "relation" admits an elegant set theoretical formulation.

Definition 26.1. A **binary relation** R from a set A to a set B is

$$R \subseteq A \times B$$

Notations: $(a, b) \in R$, aRb, etc. We will omit "binary" when safe. Examples:

a is a parent of b for humans

$a | b$ for integers

$a \leq b$ for reals, etc.

Comment 26.2. Mind the main difference between functions from A to B and relations from A to B. Relations are functions for which all the restrictions are off: relations are not necessarily total on A, relations are not necessarily single valued. Equivalently, functions from A to B are special type of relations R from A to B that are total: $Dom(R) = A$, and single valued: $(a, b) \in R$ and $(a, b') \in R$ yields $b = b'$.

Recall that subsets of a given finite set S can be represented by bit strings of length $n = |S|$. Likewise, a (binary) relation R from a finite set $A = \{a_1, a_2, \ldots, a_n\}$ to a finite set $B = \{b_1, b_2, \ldots, b_m\}$ as a subset of $A \times B$ can be represented by a bit string of length $|A| \cdot |B|$. Psychologically and mathematically it is convenient to represent this string in a row-column style, i.e. as a bit matrix $[\sigma_{i,j}]$ having dimensions $|A| \times |B|$. As before, an entry 1 in position (i, j) indicates that $(a_i, b_j) \in R$, and an entry 0 there tells that $(a_i, b_j) \notin R$:

$$\sigma_{i,j} = \begin{cases} 1, & if \ (a_i, b_j) \in R \\ 0, & if \ (a_i, b_j) \notin R \end{cases}$$

For example, the relation *(state, city)* has 50 rows and very many columns. A relation *(state, capital city)* has 50×50 matrix with exactly one 1 in each row and in each column. The total number of relation from A to B is then equal to $2^{|A| \cdot |B|}$.

Definition 26.3. Relation on a set A is a relation from A to A, i.e. a subset of $A \times A$.

Example 26.4. Matrices of relations on a set Am such that $|A| = n$ are bit $n \times n$ matrices. There 2^{n^2} distinct relations on such A. For example, the identity $n \times n$ matrix

$$I_n = \begin{pmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ . & . & \cdots & . \\ 0 & 0 & \cdots & 1 \end{pmatrix}$$

encodes the identity relation $x = y$ on A. The matrices of 1's and 0's only

$$\begin{pmatrix} 1 & 1 & \cdots & 1 \\ 1 & 1 & \cdots & 1 \\ . & . & \cdots & . \\ 1 & 1 & \cdots & 1 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 0 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ . & . & \cdots & . \\ 0 & 0 & \cdots & 0 \end{pmatrix}$$

stand for the full relation A^2 and the empty relation respectively.

Definition 26.5. A relation R on A is **reflexive**, if $(a, a) \in R$ for all $a \in A$. Examples: $x \leq y$, $n|m$, $n \equiv m$ **mod** 5, etc. Non reflexive relations: $x < y$, a is a parent of b, etc. Matrices of reflexive relations are the ones having 1's on their main diagonal. The total number of reflexive relations is then equal to the total number of such bit matrices. Since the main diagonal is occupied with 1's one has to fill in the remaining $n^2 - n$ positions arbitrarily by bits, which can be done in 2^{n^2-n} ways.

Definition 26.6. A relation R on A is **symmetric** if

$$(a, b) \in R \ \Rightarrow \ (b, a) \in R \quad \text{(equivalently, } aRb \ \Rightarrow \ bRa)$$

Examples of symmetric relations: "a is a sibling of b", a has a joint paper with b, $n \equiv m$ **mod** 5, etc. Non symmetric: $x \leq y$, $n|m$, a is a parent of b, etc. The matrices of symmetric relations are symmetric with respect to the main diagonal: $\sigma_{i,j} = \sigma_{j,i}$. The total number of such matrices can be evaluated as follows. We are free to fill in all entries on the main diagonal and above (the total of $1 + 2 + 3 + \ldots + n = n(n+1)/2$ bit entries), which can be done in $2^{n(n+1)/2}$ ways. The rest of a matrix (i.e. the area below the main diagonal) is the mirror image of the upper half, and thus is completely determined. The number of symmetric relations is $2^{n(n+1)/2}$.

Definition 26.7. A relation R on A is **antisymmetric** if aRb & $bRa \ \Rightarrow \ a = b$. Examples: $x \leq y$, $x < y$, $n|m$, "a is a parent of b", etc. The empty and identity relations are both symmetric and antisymmetric. Matrices of antisymmetric relations have arbitrary bits on the main diagonal, and not more than one 1 at each pair of symmetric non-diagonal entries. We leave as a useful exercise to evaluate the total number of antisymmetric relations on an n-element set.

Definition 26.8. A relation R on A is **transitive** if aRb & $bRc \ \Rightarrow \ aRc$. Examples: $x \leq y$, $x < y$, $n|m$, $n \equiv m$ **mod** 5, $X \subseteq Y$, $\emptyset$, etc. Non-transitive relations: "a is a parent of b, "a has a joint paper with b", $X \in Y$, etc.

Definition 26.9. An **equivalence relation** on A is a reflexive, symmetric and transitive relation on A. Examples: $a = b$, $n \equiv m$ **mod** 5, etc. We will discuss the equivalence relations in details later.

Definition 26.10. Operations on relations. Set theoretical $R_1 \cup R_2$, $R_1 \cap R_2$, $R_1 - R_2$, composition: $S \circ R = \{(a, c) \mid \exists b \in B : aRb \ \& \ bSc\}$, which is a natural generalization of the notion of composition of functions. Example, $R = S = $"$a$ is a parent of b", $R \circ S = R^2 = $"$a$ is a grandparent of b". As before, $R^0 = id$, $R^1 = R$, $R^{n+1} = R^n \circ R$.

Definition 26.11. n-ary relation is a subset of $A_1 \times A_2 \times \ldots \times A_n$. Example: ternary relation "r is a remainder of n divided by q". The standard way to represent a finite n-ary relation R is to list all n-tuples $(a_1, a_2, \ldots, a_n)$ (called **records**) in R. Such listings are called **tables**. See examples in 6.2.

Homework assignments. (The second installment due Friday 04/06)
26A:Rosen6.1-4; 26B:Rosen6.1-8; 26C:Rosen6.1-34; 26D:Rosen6.2-4cd; 26E:Rosen6.2-10.