HW 6.5: 2 14b 22b,d
6.6: 2b 10ae 24

LEC.28 11/03/99

EQUIVALENCE RELATIONS

def EQUIVALENCE RELATION ON A

REFLEXIVE, SYMMETRIC AND TRANSITIVE.

$(a,b) \in R$ ~ "a AND b ARE EQUIVALENT."

EXAMPLES. $a = b$, $a \equiv b \pmod{m}$,

a IS A RELATIVE OF b

$[a]_R = \{ b \mid (a,b) \in R \}$ THE EQUIVALENCE CLASS OF a.

(b:) A REPRESENTATIVE OF THE EQUIVALENCE CLASS

$[a]_= = \{a\}$

$[2]_{\bmod 5} = \{ \ldots -13, -8, -3, 2, 7, 12, 17, \ldots \}$

th. $[a] \cap [b] \neq \emptyset \Rightarrow [a] = [b]$

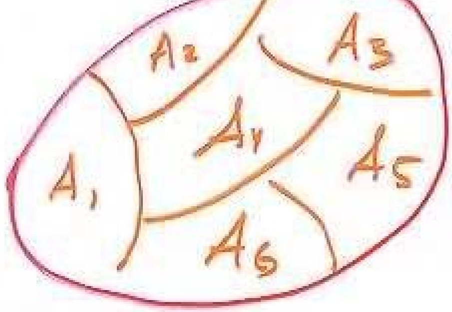
$x \in [a] \Leftrightarrow xRa \Leftrightarrow xRc \Leftrightarrow xRb \Leftrightarrow x \in [b]$

PARTITION OF S IS A COLLECTION $A_i \subseteq S$, $A_i \neq \emptyset$, $i \in I$

SUCH THAT

1. $A_i \cap A_j = \emptyset$, WHEN $i \neq j$

2. $\bigcup_{i \in I} A_i = S$



Th. EVERY PARTITION SPECIFIES AN EQUIVALENCE RELATION $aRb \iff a,b \in A_i$ FOR SOME i

R IS REFLEXIVE: aRa, SINCE $a \in A_i$ FOR SOME i

R IS SYMMETRIC: $aRb \Rightarrow a \in A_i \wedge b \in A_i$ FOR SOME i
$\Rightarrow b \in A_i \wedge a \in A_i \Rightarrow bRa$

R IS TRANSITIVE:
$$\left.\begin{array}{l} aRb \Rightarrow a \in A_i \wedge b \in A_i \\ bRc \Rightarrow b \in A_j \wedge c \in A_j \end{array}\right\} \Rightarrow A_i = A_j$$
$$\Downarrow$$
$$a \in A_i \wedge c \in A_i \Rightarrow aRc$$

FOR EVERY EQUIVALENCE RELATION ITS EQUIVALENCE CLASSES FORM A PARTITION

$[a] \neq \emptyset$, $[a] \cap [b] = \emptyset$ (OR $[a] = [b]$), $a \in [a] \Rightarrow S = \bigcup_{a \in S} [a]$

EXAMPLE. EQUIVALENCE CLASSES (MOD 3)

$[0] = \{\ldots -9, -6, -3, 0, 3, 6, 9 \ldots\}$

$[1] = \{\ldots -8, -5, -2, 1, 4, 7, 10 \ldots\}$

$[2] = \{\ldots, -7, -4, -1, 2, 5, 8, 11, \ldots\}$

$\mathbb{Z} = [0] \cup [1] \cup [2]$

PARTIAL ORDERINGS $\overset{def}{\iff}$ REFLEXIVE
ANTISYMMETRIC
TRANSITIVE

(S, R) - PARTIALLY ORDERED SET | USUAL NOTATION $aRb \sim a \preccurlyeq b$

R IS A PARTIAL ORDER(ING) ON S

EXAMPLES: $x \leq y$ (ON INTEGERS, RATIONALS, REALS)

$X \subseteq Y$, $X, Y \subseteq S$

$x | y$, x, y ARE POSITIVE INTEGERS

$a, b \in S$ ARE COMPARABLE IF EITHER $a \preccurlyeq b$ OR $b \preccurlyeq a$

$(S, \preccurlyeq)$ IS A LINEAR ORDER (TOTAL ORDER) IF EVERY TWO ELEMENTS OF S ARE COMPARABLE.

$(S, \preceq)$ IS <u>WELL-ORDERED</u> IF IT IS PARTIALLY ORDERED AND EVERY NONEMPTY SUBSET OF S HAS A LEAST ELEMENT

EXAMPLES. $(\{0,1\}, \subseteq)$ NOT A LINEAR ORDER

$a \leq b$ (ON $\mathbb{Z}, \mathbb{Q}, \mathbb{R}$) IS A LINEAR ORDER (BUT NOT WELL-ORDERED)

$(\mathbb{Z}^+, \leq)$ IS WELL-ORDERED !

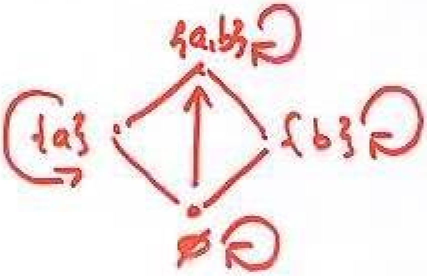
$\mathbb{Z}^+ \times \mathbb{Z}^+$ WITH LEXICOGRAPHIC ORDIRING

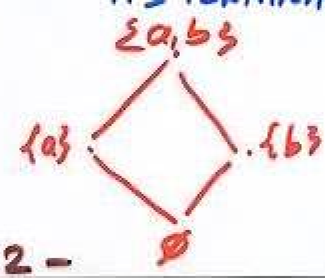
$$(a_1, a_2) \preceq (b_1, b_2) \iff a_1 < b_1 \text{ OR } a_1 = b_1 \wedge a_2 < b_2$$

IS WELL ORDERED.

<u>HASSE DIAGRAMS</u> FOR POSETS (REDUNDANT ARROWS ARE REMOVED)

1. LOOPS
2. TRANSITIVITY REDUNDANCIES
3. ARROWS, BY ARRANGING AN INITIAL VERTEX BELOW ITS TERMINAL VERTEX

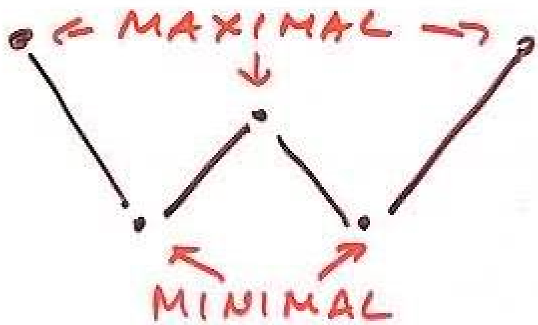




a IS A <u>MAXIMAL ELEMENT</u> IN $(S, \preccurlyeq)$ IF
THERE IS NO $b \in S$ SUCH THAT $a \prec b$

a IS <u>MINIMAL</u> IN $(S, \preccurlyeq)$ IF
THERE IS NO $b \in S$ SUCH THAT $b \prec a$

TOPS AND BOTTOMS IN HASSE DIAGRAMS



u IS <u>A UPPER BOUND</u> of A IF $a \preccurlyeq u$ FOR ALL $a \in A$

x IS <u>THE LEAST UPPER BOUND</u> OF A IF
X IS THE UPPER BOUND THAT IS LESS THAN EVERY OTHER UPPER BOUND

UNIQUE, IF ANY

LOWER BOUNDS - DUAL DEFINITION!