HW 6.5: 2 14b 22b,d
6.6: 2b 10a,c 24

LEC.28 11/03/99

EQUIVALENCE RELATIONS

def EQUIVALENCE RELATION ON A
REFLEXIVE, SYMMETRIC AND TRANSITIVE.

$(a,b) \in R \sim$ "a AND b ARE EQUIVALENT"

EXAMPLES. $a = b$, $a \equiv b \pmod m$,
a IS A RELATIVE OF b

$[a]_R = \{ b \mid (a,b) \in R \}$ THE EQUIVALENCE CLASS OF a.

A REPRESENTATIVE OF THE EQUIVALENCE CLASS

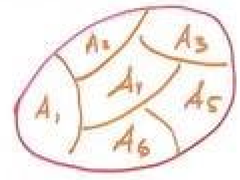
$[a]_= = \{a\}$

$[2]_{\text{mod } 5} = \{\ldots -13, -8, -3, 2, 7, 12, 17, \ldots\}$

th. $[a] \cap [b] \neq \emptyset \Rightarrow [a] = [b]$

$x \in [a] \Leftrightarrow xRa \Leftrightarrow xRc \Leftrightarrow xRb \Leftrightarrow x \in [b]$

PARTITION OF S IS A COLLECTION $A_i \subseteq S$, $A_i \neq \emptyset$, $i \in I$
SUCH THAT
1. $A_i \cap A_j = \emptyset$, WHEN $i \neq j$
2. $\bigcup_{i \in I} A_i = S$



Th. EVERY PARTITION SPECIFIES AN EQUIVALENCE RELATION $aRb \Leftrightarrow a, b \in A_i$ FOR SOME i

R IS REFLEXIVE: aRa, SINCE $a \in A_i$ FOR SOME i

R IS SYMMETRIC: $aRb \Rightarrow a \in A_i \wedge b \in A_i$ FOR SOME i $\Rightarrow b \in A_i \wedge a \in A_i \Rightarrow bRa$

R IS TRANSITIVE: $\left.\begin{array}{l} aRb \Rightarrow a \in A_i \wedge b \in A_i \\ bRc \Rightarrow b \in A_j \wedge c \in A_j \end{array}\right\} \Rightarrow A_i = A_j$
$\Downarrow$
$a \in A_i \wedge c \in A_i \Rightarrow aRc$

FOR EVERY EQUIVALENCE RELATION ITS EQUIVALENCE CLASSES FORM A PARTITION

$[a] \neq \emptyset$, $[a] \cap [b] = \emptyset$ (OR $[a]=[b]$), $a \in [a] \Rightarrow S = \bigcup_{a \in S} [a]$

EXAMPLE. EQUIVALENCE CLASSES (MOD 3)

$[0] = \{\ldots -9, -6, -3, 0, 3, 6, 9 \ldots\}$

$[1] = \{\ldots -8, -5, -2, 1, 4, 7, 10 \ldots\}$

$[2] = \{\ldots, -7, -4, -1, 2, 5, 8, 11, \ldots\}$

$\mathbb{Z} = [0] \cup [1] \cup [2]$

PARTIAL ORDERINGS $\overset{def}{\Longleftrightarrow}$ REFLEXIVE
ANTISYMMETRIC
TRANSITIVE

(S, R) - PARTIALLY ORDERED SET
R IS A PARTIAL ORDER(ING) ON S

USUAL NOTATION $aRb \sim a \preccurlyeq b$

EXAMPLES: $x \leq y$ (ON INTEGERS, RATIONALS, REALS)

$X \subseteq Y$, $X, Y \subseteq S$

$x \mid y$, x, y ARE POSITIVE INTEGERS

$a, b \in S$ ARE COMPARABLE IF EITHER $a \preccurlyeq b$ OR $b \preccurlyeq a$

$(S, \preccurlyeq)$ IS A LINEAR ORDER (TOTAL ORDER) IF EVERY TWO ELEMENTS OF S ARE COMPARABLE.

$(S, \preccurlyeq)$ IS WELL-ORDERED IF IT IS PARTIALLY ORDERED AND EVERY NONEMPTY SUBSET OF S HAS A LEAST ELEMENT

EXAMPLES. $(\{0,1\}, \subseteq)$ NOT A LINEAR ORDER

$a \leq b$ (ON $\mathbb{Z}$, $\mathbb{Q}$, $\mathbb{R}$) IS A LINEAR ORDER (BUT NOT WELL-ORDERED)

$(\mathbb{Z}^+, \leq)$ IS WELL-ORDERED !

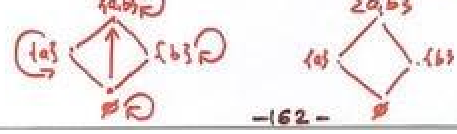
$\mathbb{Z}^+ \times \mathbb{Z}^+$ WITH LEXICOGRAPHIC ORDERING

$(a_1, a_2) \leq (b_1, b_2) \Leftrightarrow a_1 < b_1$ OR $a_1 = b_1 \wedge a_2 < b_2$

IS WELL ORDERED.

HASSE DIAGRAMS FOR POSETS (REDUNDANT ARROWS ARE REMOVED)

1. LOOPS
2. TRANSITIVITY REDUNDANCIES
3. ARROWS, BY ARRANGING AN INITIAL VERTEX BELOW ITS TERMINAL VERTEX



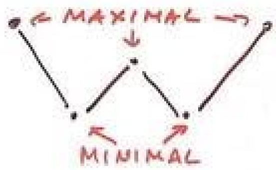
a IS A MAXIMAL ELEMENT IN $(S, \preccurlyeq)$ IF

THERE IS NO $b \in S$ SUCH THAT $a \prec b$

a IS MINIMAL IN $(S, \preccurlyeq)$ IF

THERE IS NO $b \in S$ SUCH THAT $b \prec a$

TOPS AND BOTTOMS IN HASSE DIAGRAMS



u IS A UPPER BOUND of A IF $a \preccurlyeq u$ FOR ALL $a \in A$

x IS THE LEAST UPPER BOUND OF A IF

X IS THE UPPER BOUND THAT IS LESS THAN EVERY OTHER UPPER BOUND

UNIQUE, IF ANY

LOWER BOUNDS - DUAL DEFINITION!