6.4: 20, 26a,c

LEC. 27 11/01/99

RELATIONS II

$R \subseteq A \times B$ A RELATION (BINARY) $A=\{a_1,\dots a_k\}$, $B=\{b_1,\dots b_n\}$

$M_R = [m_{ij}]$, WHERE $m_{ij} = \begin{cases} 1 & \text{IF } (a_i, b_j) \in R \\ 0 & \text{O.W.} \end{cases}$

EXAMPLES. $\begin{bmatrix} 0 & \cdots & 0 \\ 0 & \cdots & 0 \end{bmatrix}$ ~ EMPTY RELATION

$\begin{bmatrix} 1 & \cdots & 1 \\ 1 & \cdots & 1 \end{bmatrix}$ ~ THE WHOLE OF $A \times B$

$k\left\{\underbrace{\begin{bmatrix} 1 & & 0 \\ & \ddots & \\ 0 & & 1 \end{bmatrix}}_{k}\right.$ ~ THE IDENTITY RELATION ON $A \times A$

$\begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ ANTISYMMETRIC RELATIONS

\# OF ANTISYMMETRIC RELATIONS ON A:

$|A|=1$ $\# = 2$: $[0]$ $[1]$

$|A|=2$ $\# = 12$: $\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$,

$\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \cdots \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \dots \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$. $|A| = n$, $\# = ?$

REPRESENTING OPERATIONS

$M_{R \cup S} = M_R \vee M_S$

$M_{R \cap S} = M_R \wedge M_S$

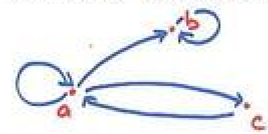
$M_{\underbrace{R \circ S}} = \underbrace{M_S \odot M_R}$

BOOLEAN PRODUCT

THE ORDER CHANGES DUE TO THE AWKWARD NOTATION IN $R \circ S$.

$M_{R^2} = M_R^2$ BOOLEAN SQUARE $\begin{bmatrix} 1 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 1 \end{bmatrix}^2 = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 1 \end{bmatrix}$

DIGRAPHS REPRESENTING RELATIONS



$V = \{a, b, c\}$ - VERTICES (NODES)

$E = \{(a,a), (a,b), (b,b), (a,c), (c,a)\}$ EDGES

(a, b) INITIAL VERTEX, TERMINAL VERTEX

(a, a) - A LOOP

R IS REFLEXIVE $\iff$ THE IS A LOOP AT EVERY VERTEX

R IS SYMMETRIC $\iff$ EVERY EDGE BETWEEN DISTINCT VERTICES HAS THE EDGE IN THE OPPOSITE DIRECTION

R IS TRANSITIVE $\iff$ "EVERY TRIANGLE IS CLOSED"

CLOSURES OF RELATIONS.

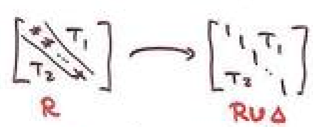
LET $\mathbb{P}$ BE A PROPERTY OF RELATIONS (REFLEXIVITY, SYMMETRY, TRANSITIVITY, ETC). A CLOSURE OF R WITH RESPECT TO $\mathbb{P}$ IS THE SMALLEST RELATION S WITH PROPERTY $\mathbb{P}$ CONTAINING R (IF ANY).

HERE "SMALLEST" MEANS

$S \subseteq X$ FOR ANY PROPERTY X WITH $\mathbb{P}$ CONTAINING R

REFLEXIVE CLOSURE OF $R = R \cup \Delta$, WHERE

$\Delta = \{(a,a) \mid a \in A\}$ (DIAGONAL OF A)

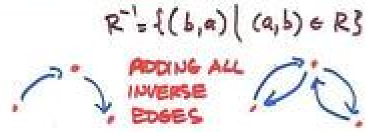


REFL. CLOSURE OF "<" IS "≤".

ADDING ALL POSSIBLE LOOPS

SYMMETRIC CLOSURE OF $R = R \cup R^{-1}$, WHERE

$R^{-1} = \{(b,a) \mid (a,b) \in R\}$



SYMM. CLOSURE OF ≤ IS ≠

A PATH IN A DIGRAPH G: A SEQUENCE OF EDGES $(x_0 x_1)(x_1 x_2)(x_2 x_3) \dots (x_{n-1} x_n)$. NOTATION $x_0 x_1 x_2 x_3 \dots x_{n-1} x_n$. // CYCLE (CIRCUIT) $x_0 x_1 \dots x_{n-1} x_0$.

n = length of THE PATH.

Th. THERE IS A PATH OF LENGTH n FROM a TO b IN R IFF $(a,b) \in R^n$

PROOF. INDUCTION ON n.

$n=1$ – BY DEFINITION

ASSUME THE THEOREM HOLDS FOR $n=k$ (I.H.)

THERE IS A PATH FROM a TO b OF LENGTH $k+1$ IFF FOR SOME c SUCH THAT $(c,b) \in R$ THERE IS A PATH FROM a TO c OF LENGTH k

BY THE I.H., $(a,c) \in R^k$,

THEREFORE $(a,b) \in R^{k+1}$

CONNECTIVITY RELATION $R^* = \{(a,b) \mid$ THERE IS A PATH FROM a TO b IN $R\}$

$$R^* = \bigcup_{n=1}^{\infty} R^n$$

EXAMPLE. $R = \{(a,b) \mid$ COMPUTER a HAD AT LEAST ONE CONNECTION WITH A NET COMPUTER $b\}$

$R^* =$ WWW

Th. THE TRANSITIVE CLOSURE OF R IS R^*.

PROOF. TR.CL.$(R) \subseteq R^*$ SINCE R^* IS TRANSITIVE

$R^* \subseteq$ TR.CL.(R) SINCE A CONNECTED PAIR (a,b) BELONGS TO ANY TRANSITIVE $S \supseteq R$

Th. $R^* = R \cup R^2 \cup \ldots \cup R^n$, WHERE $n=|A|$

R IS A RELATION ON A.

PROOF. IMMEDIATELY FOLLOWS FROM AN OBSERVATION THAT IF a AND b FROM A ARE CONNECTED IN R, THEN THERE IS A PATH FROM a TO b WITH LENGTH NOT EXCEEDING n.

INDEED, BY THE PIGEONHOLE PRINCIPLE LONGER PATHS HAVE EQUAL VERTICES, THEREFORE CYCLES. CYCLES CAN BE DELETED!

COR. $M_{R^*} = M_R \vee M_R^2 \vee \ldots \vee M_R^n$

EXAMPLE. $M_R = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 0 \end{bmatrix}$; $M_R^2 = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix}$;

$M_R^3 = M_R^2$ $\quad M_{R^*} = M_R \vee M_R^2 \vee M_R^3 = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix}$