LEC. 25 10/27/99

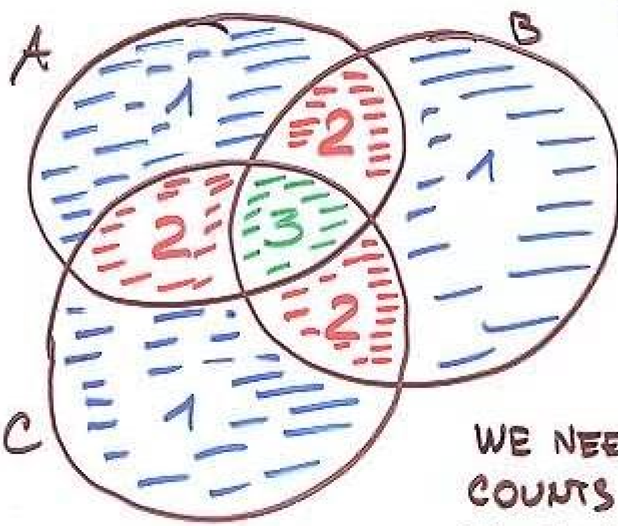
MORE ON INCLUSION - EXCLUSION

Th. $|A \cup B| = |A| + |B| - |A \cap B|$


THESE ELEMENTS ARE COUNTED TWICE IN $|A| + |B|$:

AS MEMBERS OF A AND AS MEMBERS OF B.

WHAT ABOUT $|A \cup B \cup C| = |A| + |B| + |C|$ ?



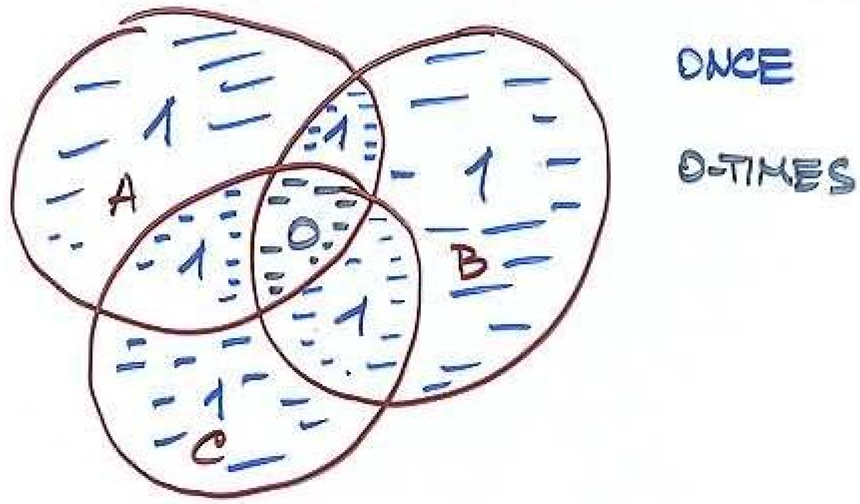

BLUE ELEMENTS ARE COUNTED ONCE

REDS ARE COUNTED TWICE

GREENS ARE COUNTED THREE TIMES

WE NEED A FORMULA THAT COUNTS EACH ELEMENT IN $A \cup B \cup C$ ONLY ONCE !

WHAT ABOUT $|A|+|B|+|C|-|A\cap B|-|A\cap C|-|B\cap C|$ ?




$|A\cup B\cup C| = |A|+|B|+|C|-|A\cap B|-|A\cap C|-|B\cap C|+|A\cap B\cap C|$

COUNTS EACH ELEMENT IN $A\cup B\cup C$ ONCE!

EXAMPLE. EVERY STUDENT IN SOME CLASS TAKES AT LEAST ONE OF THREE COURSES: MATH, PHILOSOPHY OR VINE TESTING

| | | |
|---|---|---|
| M = 30 | M&P = 10 | M&P&V = 5 |
| P = 40 | M&V = 20 | |
| V = 100 | P&V = 20 | |

HOW MANY STUDENTS ARE THERE IN CLASS?

$N = M + P + V - M\&P - M\&V - P\&V + M\&P\&V =$

$= 30+40+100-10-20-20+5 = 170-50+5 = 125$

EXAMPLE. SIMILAR SETUP, BUT NOW WE KNOW $\mathcal{N}$ (THE TOTAL NUMBER OF STUDENTS IN CLASS) $\mathcal{N} = 125$, AND ALSO

| | | |
|---|---|---|
| $M = 30$ | $M \& P = 10$ | FIND $M \& P \& U$ |
| $P = 40$ | $M \& U = 20$ | (HOW MANY |
| $V = 100$ | $P \& V = 20$ | STUDENTS TAKE ALL THREE COURSES) |

$$125 = 30 + 40 + 100 - 10 - 20 - 20 + X$$

$$125 = 170 - 50 + X \;;\quad 125 = 120 + X \;;\quad X = 5$$

**Th.** $|A_1 \cup A_2 \cup \ldots \cup A_n| = |A_1| + \ldots + |A_n| - \sum_{1 \le i < j \le n} |A_i \cap A_j|$

$$+ \sum_{1 \le i < j < k \le n} |A_i \cap A_j \cap A_k| - \quad \ldots$$

$$\ldots (-1)^{n+1} |A_1 \cap A_2 \cap \ldots \cap A_n|$$

PROOF. IT SUFFICES TO SHOW THAT ANY $a \in A_1 \cap A_2 \cap \ldots \cap A_z$, SUCH THAT $a \notin A_{z+1}, a \notin A_{z+2}, \ldots, a \notin A_n$ IS COUNTED ONCE (FOR ANY $a$, $z$).

| | | |
|---|---|---|
| $\lvert A_1 \rvert + \lvert A_2 \rvert + \ldots + \lvert A_n \rvert$ | $C(z,1)$ | TIMES |
| $- \lvert A_1 \cap A_2 \rvert - \lvert A_1 \cap A_3 \rvert - \ldots$ | $- C(z,2)$ | TIMES |
| $+ \lvert A_1 \cap A_2 \cap A_3 \rvert + \ldots$ | $+ C(z,3)$ | TIMES |
| $(-1)^{z+1} (\lvert A_1 \cap A_2 \cap \ldots \cap A_z \rvert + \ldots)$ | $(-1)^{z+1} \cdot C(z,z)$ | TIMES |
| $\ldots \lvert A_1 \cap A_2 \cap \ldots \cap A_{z+1} \rvert + \ldots$ | $0$ | TIMES |

$$x = C(z,1) - C(z,2) + C(z,3) - \ldots (-1)^{z+1} C(z,z)$$

$\uparrow$

THE NUMBER OF TIMES $a$ IS COUNTED

SINCE $\quad C(z,0) - C(z,1) + C(z,2) - \cdots (-1)^{z} C(z,z) = 0$

$1 = C(z,0) = x$

EXAMPLE. $n=4$

$$|A \cup B \cup C \cup D| = |A| + |B| + |C| + |D|$$
$$- |A \cap B| - |A \cap C| - |A \cap D| - |B \cap C| - |B \cap D| - |C \cap D|$$
$$+ |A \cap B \cap C| + |A \cap B \cap D| + |A \cap C \cap D| + |B \cap C \cap D|$$
$$- |A \cap B \cap C \cap D|$$

$n=5$ $|A_1 \cup A_2 \cup A_3 \cup A_4 \cup A_5| = |A_1| + |A_2| + |A_3| + |A_4| + |A_5|$

$- |A_1 \cap A_2| - |A_1 \cap A_3| - |A_1 \cap A_4| - |A_1 \cap A_5| - |A_2 \cap A_3| - |A_2 \cap A_4| -$
$- |A_2 \cap A_5| - |A_3 \cap A_4| - |A_3 \cap A_5| - |A_4 \cap A_5|$

$+ |A_1 \cap A_2 \cap A_3| + \cdots + |A_3 \cap A_4 \cap A_5|$

$- |A_1 \cap A_2 \cap A_3 \cap A_4| - |A_1 \cap A_2 \cap A_3 \cap A_5| - \ldots |A_2 \cap A_3 \cap A_4 \cap A_5|$

$+ |A_1 \cap A_2 \cap A_3 \cap A_4 \cap A_5|$

$2^n - 1$ TERMS TO THE RIGHTHANDSIDE.