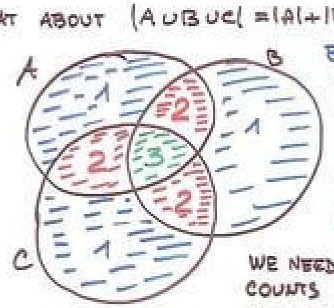
LEC. 25 10/27/99

MORE ON INCLUSION - EXCLUSION

Th. $|A \cup B| = |A| + |B| - |A \cap B|$


THESE ELEMENTS ARE COUNTED TWICE IN $|A|+|B|$: AS MEMBERS OF A AND AS MEMBERS OF B.

WHAT ABOUT $|A \cup B \cup C| = |A| + |B| + |C|$ ?




BLUE ELEMENTS ARE COUNTED ONCE

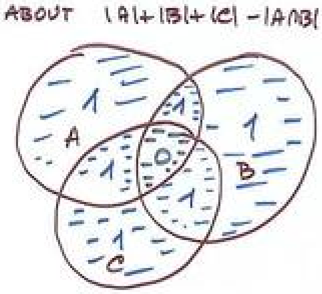
REDS ARE COUNTED TWICE

GREENS ARE COUNTED THREE TIMES

WE NEED A FORMULA THAT COUNTS EACH ELEMENT IN $A \cup B \cup C$ ONLY ONCE!



WHAT ABOUT $|A|+|B|+|C| - |A \cap B| - |A \cap C| - |B \cap C|$ ?




ONCE

0-TIMES

$|A \cup B \cup C| = |A|+|B|+|C| - |A \cap B| - |A \cap C| - |B \cap C| + |A \cap B \cap C|$
COUNTS EACH ELEMENT IN $A \cup B \cup C$ ONCE!

EXAMPLE. EVERY STUDENT IN SOME CLASS TAKES AT LEAST ONE OF THREE COURSES: MATH, PHILOSOPHY OR VINE TESTING

| | | |
|---|---|---|
| $M = 30$ | $M \& P = 10$ | $M \& P \& V = 5$ |
| $P = 40$ | $M \& V = 20$ | |
| $V = 100$ | $P \& V = 20$ | HOW MANY STUDENTS ARE THERE IN CLASS? |

$N = M + P + V - M\&P - M\&V - P\&V + M\&P\&V =$
$= 30 + 40 + 100 - 10 - 20 - 20 + 5 = 170 - 50 + 5 = 125$



EXAMPLE. SIMILAR SETUP, BUT NOW WE KNOW $N$ (THE TOTAL NUMBER OF STUDENTS IN CLASS) $N = 125$, AND ALSO

| | | |
|---|---|---|
| $M = 30$ | $M\&P = 10$ | FIND $M\&P\&V$ |
| $P = 40$ | $M\&V = 20$ | (HOW MANY STUDENTS TAKE ALL THREE COURSES) |
| $V = 100$ | $P\&V = 20$ | |

$125 = 30 + 40 + 100 - 10 - 20 - 20 + X$

$125 = 170 - 50 + X$; $125 = 120 + X$; $X = 5$

Th. $|A_1 \cup A_2 \cup \ldots \cup A_n| = |A_1| + \ldots + |A_n| - \sum_{1 \le i < j \le n} |A_i \cap A_j| + \sum_{1 \le i < j < k \le n} |A_i \cap A_j \cap A_k| - \ldots$

$\ldots (-1)^{n+1} |A_1 \cap A_2 \cap \ldots \cap A_n|$

PROOF. IT SUFFICES TO SHOW THAT ANY $a \in A_1 \cap A_2 \cap \ldots \cap A_t$, SUCH THAT $a \notin A_{t+1}, a \notin A_{t+2}, \ldots, a \notin A_n$ IS COUNTED ONCE (FOR ANY $a$, $t$).

| | |
|---|---|
| $\lvert A_1\rvert + \lvert A_2\rvert + \ldots + \lvert A_n\rvert$ | $C(t,1)$ TIMES |
| $-\lvert A_1 \cap A_2\rvert - \lvert A_1 \cap A_3\rvert - \ldots$ | $-C(t,2)$ TIMES |
| $+\lvert A_1 \cap A_2 \cap A_3\rvert + \ldots$ | $+C(t,3)$ TIMES |
| $(-1)^{t+1}(\lvert A_1 \cap A_2 \cap \ldots \cap A_t\rvert + \ldots)$ | $(-1)^{t+1} C(t,t)$ TIMES |
| $\ldots \lvert A_1 \cap A_2 \cap \ldots \cap A_{t+1}\rvert + \ldots$ | $0$ TIMES |



$x = C(t,1) - C(t,2) + C(t,3) - \ldots (-1)^{t+1} C(t,t)$

THE NUMBER OF TIMES $a$ IS COUNTED

SINCE $C(t,0) - C(t,1) + C(t,2) - \ldots (-1)^t C(t,t) = 0$

$1 = C(t,0) = x$

EXAMPLE. $n=4$

$|A \cup B \cup C \cup D| = |A|+|B|+|C|+|D|$
$-|A \cap B| - |A \cap C| - |A \cap D| - |B \cap C| - |B \cap D| - |C \cap D|$
$+|A \cap B \cap C| + |A \cap B \cap D| + |A \cap C \cap D| + |B \cap C \cap D|$
$-|A \cap B \cap C \cap D|$

$n=5$ $|A_1 \cup A_2 \cup A_3 \cup A_4 \cup A_5| = |A_1| + |A_2| + |A_3| + |A_4| + |A_5|$
$-|A_1 \cap A_2| - |A_1 \cap A_3| - |A_1 \cap A_4| - |A_1 \cap A_5| - |A_2 \cap A_3| - |A_2 \cap A_4| - |A_2 \cap A_5| - |A_3 \cap A_4| - |A_3 \cap A_5| - |A_4 \cap A_5|$
$+|A_1 \cap A_2 \cap A_3| + \ldots + |A_3 \cap A_4 \cap A_5|$
$-|A_1 \cap A_2 \cap A_3 \cap A_4| - |A_1 \cap A_2 \cap A_3 \cap A_5| - \ldots |A_2 \cap A_3 \cap A_4 \cap A_5|$
$+|A_1 \cap A_2 \cap A_3 \cap A_4 \cap A_5|$

$2^n - 1$ TERMS TO THE RIGHTHAND SIDE.