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RECURRENCE SEQUENCES

EXAMPLE. THE SIZE OF A CERTAIN FISH POPULATION IN CAYUGA LAKE CAN INCREASE 10% A YEAR DUE TO NATURAL GROWTH. THE HARVESTING RATE IS 1000 INDIVIDUAL PER YEAR. IF THE INITIAL POPULATION SIZE IS 8000 INDIVIDUALS FIND THE POPULATION SIZE AFTER 5 YEARS.

X_n = THE POPULATION SIZE AFTER n YEARS.

$$\underbrace{X_0 = 8000;}_{\text{INITIAL CONDITION}} \quad \underbrace{X_{n+1} = 1.1 \cdot X_n - 1000}_{\text{RECURRENCE RELATION}}$$

$$\left.\begin{aligned} X_1 &= 1.1 \cdot X_0 - 1000 = 8800 - 1000 = 7800 \\ X_2 &= 1.1 \cdot X_1 - 1000 = 7580 \\ X_3 &= 1.1 \cdot X_2 - 1000 = 7338 \\ X_4 &= 1.1 \cdot X_3 - 1000 \approx 7072 \\ X_5 &= 1.1 \cdot X_4 - 1000 \approx 6779 \\ &\dots \end{aligned}\right\} \text{SOLUTION OF THE RECURRENCE RELATION (SEQUENCE).}$$

RECURRENCE RELATION	SOLUTION
$x_{n+1} = x_n + d$	$x_0, x_0+d, \dots, x_0+(n-1)d, \dots$ ARITHMETIC PROGRESSION
$x_{n+1} = x_n \cdot q$	$x_0, x_0 q, \dots, x_0 q^{n-1}$ GEOMETRIC PROGRESSION
$x_{n+2} = x_n + x_{n+1}$, $x_0 = 0, x_1 = 1$	$0, 1, 1, 2, 3, 5, 8, 13, \dots f_n, \dots$ FIBONACCI NUMBERS

COMPOUND INTEREST. THE INITIAL DEPOSIT IS $10000 AT A BANK YIELDING 5% PER YEAR WITH INTEREST COMPOUNDED ANNUALLY. HOW MUCH WILL BE THE AMOUNT AFTER n YEARS?

$S_0 = 10\,000$

$S_n = 1.05 \cdot S_{n-1}$

$S_0 = 10000$

$S_1 = 1.05 \cdot S_0 = 1.05 \cdot 10\,000$

$S_2 = 1.05 \cdot S_1 = (1.05)^2 S_0$

$S_n = 1.05 \cdot S_{n-1} = (1.05)^n S_0$

$S_{10} = 16\,288.95$

$S_{30} = 315\,000$ $\quad S_{100} = 1\,315\,012.6$

NUMBER OF BIT STRINGS THAT DO NOT HAVE TWO CONSECUTIVE 0s.

$a_n = X + Y$, X = # OF n-STRINGS ENDING WITH 1. EACH OF THEM IS AN $(n-1)$ STRING WITH NO CONSECUTIVE 0s WITH A 1 ADDED

$X = a_{n-1}$

Y = # ON n-STRINGS ENDING WITH 0. EACH IS AN $(n-2)$ STRING WITH NO CONSECUTIVE 0s WITH 10 ADDED.

$Y = a_{n-2}$

$$a_n = a_{n-1} + a_{n-2}$$

$a_1 = 2$; $a_2 = 3$ - THE INITIAL CONDITION

$a_4 = 5$, $a_5 = 8$, $a_6 = 13, \dots$ LIKE f_n: $a_n = f_{n+2}$

CODEWORD ENUMERATION. CODEWORD = STRING OF DECIMAL DIGITS WITH EVEN # OF 0'S

a_n = # OF CODEWORDS OF LENGTH n.

$a_1 = 9$;

DISJOINT CASES!

1° VALID C.W. $n-1$ + NONZERO = VALID C.W. n

2° INVALID C.W. $n-1$ + 0 = VALID C.W. n

$$a_n = a_{n-1} \cdot 9 + (10^{n-1} - a_{n-1}) \cdot 1 = 8a_{n-1} + 10^{n-1}$$

EXAMPLE. MESSAGES ARE TRANSMITTED THROUGH A COMMUNICATIONS CHANNEL USING TWO SIGNALS ONE REQUIRES 1 MICROSECOND, THE OTHER 2 M.S.

a_n = THE # OF DIFFERENT MESSAGES THAT CAN BE SEND IN n MICROSECONDS. (NO BLANKS)

$a_0 = 1$, $a_1 = 1$. EACH SIGNAL OF THE LENGTH n IS OBTAINED BY ONE AND ONLY ONE OF THE FOLLOWING TWO PROCEDURES:

1° $\underbrace{\boxed{\quad\quad}}_{n-1} \cdot \underbrace{\blacksquare}_{1} = \underbrace{\boxed{\quad\quad\quad}}_{n}$

2° $\underbrace{\boxed{\quad\quad}}_{n-2} \cdot \underbrace{\blacksquare\blacksquare}_{2} = \underbrace{\boxed{\quad\quad\quad}}_{n}$

$$\left. \begin{aligned} &a_n = a_{n-1} + a_{n-2}\,; \quad a_0 = 1, \; a_1 = 1 \\ &1, 1, 2, 3, 5, 8, 13, \dots \end{aligned} \right\} \; a_n = f_{n+1}$$

PARENTHESIZING PRODUCTS.

$x_0 \cdot x_1 \cdot x_2 \to x_0 \cdot (x_1 \cdot x_2)$, $(x_0 \cdot x_1) \cdot x_2$

$x_0 \cdot x_1 \cdot x_2 \cdot x_3 \to x_0 \cdot (x_1 \cdot (x_2 \cdot x_3))$, $x_0 \cdot ((x_1 \cdot x_2) \cdot x_3)$, $(x_0 \cdot (x_1 \cdot x_2)) \cdot x_3$, $(x_0 \cdot x_1) \cdot (x_2 \cdot x_3)$, $((x_0 \cdot x_1) \cdot x_2) \cdot x_3$

$C_0 = 1, C_1 = 1, C_2 = 2, C_3 = 5, \dots$ CATALAN NUMBERS

$x_0 \cdot x_1 \cdot x_2 \cdot \dots \cdot x_{n-1} \cdot x_n$ PICK THE OUTERMOST MULTIPLICATION. IT BREAKS THE PROBLEM OF SIZE n INTO TWO OF THE COMBINED SIZE $n-1$

$$C_n = C_0 C_{n-1} + C_1 C_{n-2} + \dots + C_{n-1} C_0$$