

HW 4.6: 8 12 18 38

LEC. 22 10/20/99

PERMUTATIONS AND COMBINATIONS II.

Th. THE NUMBER OF n -PERMUTATIONS OF n OBJECTS WITH REPETITIONS IS n^n

PROOF. n POSITIONS, n VARIANTS FOR EACH.
BY THE PRODUCT RULE $\# = \underbrace{n \cdot n \cdots n}_{n \text{ TIMES}} = n^n$

EXAMPLES 0-1 STRINGS OF LENGTH k : 2^k
WORDS IN ENGLISH OF LENGTH 3: 26^3

SAMPLING WITH REPLACEMENT.

EXAMPLE. AN URN CONTAINS 3 BLUE BALLS AND 5 RED ONES. WHAT IS THE PROBABILITY OF DRAWING 4 BLUE BALLS IN A ROW?

8 BALLS TOTAL, 8^4 OUTCOMES, 3 BLUE BALLS
 3^4 SUCCESSFUL OUTCOMES $p = \frac{3^4}{8^4} = \left(\frac{3}{8}\right)^4$

COMBINATIONS WITH REPETITION.

EXAMPLE UNLIMITED SUPPLY OF APPLES, ORANGES AND PEARS. HOW MANY 4-COMBINATIONS ARE THERE?

BRUTE FORCE: 4A, 4O, 4P, 3A1O, 3A1P, 3O1A, 3O1P, 3P1A, 3P1O, 2A2O, 2A2P, 2O2P, 2A1O1P, 2O1A1P, 2P1A1O = 15

GENERAL METHOD. EACH COMBINATION CAN BE REPRESENTED BY A STRING OF TWO BARS AND FOUR STARS


 MARKED BY *'S.

4A *****| |

2A 10 1P **|*|*

*||*** $\longleftrightarrow$ 1A 3P

THE TOTAL # OF SUCH STRINGS IS $C(4+2, 2) = C(6, 2)$

$$= \frac{6!}{4! \cdot 2!} = \frac{6 \cdot 5}{2} = 3 \cdot 5 = 15$$

TH. THERE ARE $C(n+z-1, z)$ z -COMBINATIONS FROM n ELEMENTS WITH REPETITIONS.

PROOF. EACH SUCH COMBINATION IS REPRESENTED BY A STRING OF $n-1$ BARS AND z STARS.

THERE ARE $C(n-1+z, z)$ SUCH STRINGS.

EXAMPLE FIVE BILLS FROM A BAG WITH \$1, \$5, \$10, \$20, \$50 AND \$100 BILLS.

OF COMBINATIONS? $z=5, n=6$

$$C(6+5-1, 5) = C(10, 5) = \frac{10!}{5! \cdot 5!}$$

EXAMPLE # OF SOLUTIONS OF $x_1 + x_2 + x_3 = 100$
IN NONNEGATIVE INTEGERS x_1, x_2, x_3 .

$$\begin{aligned} n=3, \quad z=100 \quad C(3+100-1, 100) &= \\ &= C(102, 100) = C(102, 2) = \\ &= \frac{102 \cdot 101}{2} = 101 \cdot 51 \end{aligned}$$

PERMUTATIONS OF SETS WITH INDISTINGUISHABLE OBJECTS

EXAMPLE. # OF DIFFERENT PERMUTATIONS OF THE
LETTERS OF THE WORD CORNELL

TOTAL # OF PERMUTATIONS OF 7 ENTRIES IS $7!$

HERE EACH STRING WAS COUNTED TWICE

$$L_1 \text{ CORNEL}_2 \sim L_2 \text{ CORNEL}_1$$

THE FINAL COUNT IS $\frac{7!}{2!}$

TH. THE NUMBER OF DIFFERENT PERMUTATIONS OF
 n OBJECTS WITH $n_1, n_2, \dots, n_k$ INDISTINGUISHABLE
OBJECTS OF TYPES $1, 2, \dots, k$ RESPECTIVELY IS

$$\frac{n!}{n_1! n_2! \dots n_k!} \quad (*)$$

PROOF. IF ALL OBJECTS WERE DISTINCT THEN
WE'D HAVE $n!$ PERMUTATIONS. SINCE n_i OBJECTS OF
TYPE i ARE INDISTINGUISHABLE WE COUNTED EACH PERMUT-
ATION $n_i!$ TIMES. THE FINAL COUNT IS $(*)$.

DISTRIBUTING OBJECTS INTO BOXES

(BOTH OBJECTS AND BOXES ARE DISTINGUISHABLE)

EXAMPLE # OF WAYS TO DISTRIBUTE HANDS OF 5 CARDS TO EACH OF 4 PLAYERS FROM THE STANDARD DECK OF 52 CARDS.

HAND1. $C(52, 5)$ COMBINATIONS

HAND2 (WHEN HAND1 CHOSEN) $C(47, 5)$

HAND3 (H1, H2 CHOSEN) $C(42, 5)$

HAND4 (H1, H2, H3 CHOSEN) $C(37, 5)$

BY THE PRODUCT RULE: $\# = C(52, 5) \cdot C(47, 5) \cdot C(42, 5) \cdot C(37, 5) =$

$$= \frac{52!}{5! \cdot 47!} \cdot \frac{47!}{5! \cdot 42!} \cdot \frac{42!}{5! \cdot 37!} \cdot \frac{37!}{5! \cdot 32!} = \frac{52!}{5! \cdot 5! \cdot 5! \cdot 5! \cdot 32!}$$

THE SECOND SOLUTION: $\# =$ NUMBER OF PERMUTATIONS
WITH 5, 5, 5, 5, 32
INDISTINGUISHABLE OBJECTS

INDEED, THE TYPES OF 52 CARDS CORRESPOND TO
HANDS 1, 2, 3, 4 (FIVE CARDS OF EACH TYPE) AND
THE REMAINING SET (32 CARDS)

$$\# = \frac{52!}{5! \cdot 5! \cdot 5! \cdot 5! \cdot 32!}$$

TH. THE NUMBER OF WAYS TO DISTRIBUTE n DISTINCT
OBJECTS INTO k DISTINCT BOXES $n_1, \dots, n_k$ OBJECTS
EACH IS

$$\frac{n!}{n_1! \cdot n_2! \cdot \dots \cdot n_k!}$$