HW 4.6: 8 12 18 38

LEC. 22 10/20/99

PERMUTATIONS AND COMBINATIONS II.

Th. THE NUMBER OF r-PERMUTATIONS OF n OBJECTS WITH REPETITIONS IS n^r

PROOF. r POSITIONS, n VARIANTS FOR EACH. BY THE PRODUCT RULE $\# = \underbrace{n \cdot n \cdots n}_{r \text{ TIMES}} = n^r$

EXAMPLES 0-1 STRINGS OF LENGTH k: 2^k
WORDS IN ENGLISH OF LENGTH 3: 26^3

SAMPLING WITH REPLACEMENT.
EXAMPLE. AN URN CONTAINS 3 BLUE BALLS AND 5 RED ONES. WHAT IS THE PROBABILITY OF DRAWING 4 BLUE BALLS IN A ROW?

8 BALLS TOTAL, 8^4 OUTCOMES, 3 BLUE BALLS
3^4 SUCCESSFUL OUTCOMES $p = \frac{3^4}{8^4} = \left(\frac{3}{8}\right)^4$

COMBINATIONS WITH REPETITION.
EXAMPLE UNLIMITED SUPPLY OF APPLES, ORANGES AND PEARS. HOW MANY 4-COMBINATIONS ARE THERE?
BRUTE FORCE: 4A, 4O, 4P, 3A1O, 3A1P, 3O1A, 3O1P, 3P1A, 3P1O,
2A2O, 2A2P, 2O2P,
2A1O1P, 2O1A1P, 2P1A1O $= 15$

GENERAL METHOD. EACH COMBINATION CAN BE REPRESENTED BY A STRING OF TWO BARS AND FOUR STARS

$\underbrace{\qquad}_{\#\text{ APPLES}} | \underbrace{\qquad}_{\#\text{ ORANGES}} | \underbrace{\qquad}_{\#\text{ PEARS}}$ MARKED BY *'S.

4A ****||
2A1O1P **|*|*
*||*** ⟼ 1A3P

THE TOTAL # OF SUCH STRINGS IS $C(4+2,2) = C(6,2)$
$= \frac{6!}{4!\cdot 2!} = \frac{6\cdot 5}{2} = 3\cdot 5 = 15$

Th. THERE ARE $C(n+r-1, r)$ r-COMBINATIONS FROM n ELEMENTS WITH REPETITIONS.

PROOF. EACH SUCH COMBINATION IS REPRESENTED BY A STRING OF n-1 BARS AND r STARS.
THERE ARE $C(n-1+r, r)$ SUCH STRINGS.

EXAMPLE FIVE BILLS FROM A BAG WITH \$1, \$5, \$10, \$20, \$50 AND \$100 BILLS.
\# OF COMBINATIONS? $r=5$, $n=6$
$C(6+5-1, 5) = C(10,5) = \frac{10!}{5!\cdot 5!}$

EXAMPLE # OF SOLUTIONS OF $x_1+x_2+x_3=100$ IN NONNEGATIVE INTEGERS x_1, x_2, x_3.
$n=3$, $r=100$ $C(3+100-1, 100) =$
$= C(102,100) = C(102,2) =$
$= \frac{102\cdot 101}{2} = 101\cdot 51$

PERMUTATIONS OF SETS WITH INDISTINGUISHABLE OBJECTS

EXAMPLE. # OF DIFFERENT PERMUTATIONS OF THE LETTERS OF THE WORD CORNELL

TOTAL # OF PERMUTATIONS OF 7 ENTRIES IS 7!
HERE EACH STRING WAS COUNTED TWICE
$L_1CORNEL_2 \sim L_2CORNEL_1$
THE FINAL COUNT IS $\frac{7!}{2!}$

Th. THE NUMBER OF DIFFERENT PERMUTATIONS OF n OBJECTS WITH $n_1, n_2, \ldots, n_k$ INDISTINGUISHABLE OBJECTS OF TYPES 1, 2, ... k RESPECTIVELY IS

$$\frac{n!}{n_1!\, n_2! \cdots n_k!} \qquad (*)$$

PROOF. IF ALL OBJECTS WERE DISTINCT THEN WE'D HAVE n! PERMUTATIONS. SINCE n_i OBJECTS OF TYPE i ARE INDISTINGUISHABLE WE COUNTED EACH PERMUTATION $n_i!$ TIMES. THE FINAL COUNT IS (*).

DISTRIBUTING OBJECTS INTO BOXES
(BOTH OBJECTS AND BOXES ARE DISTINGUISHABLE)

EXAMPLE # OF WAYS TO DISTRIBUTE HANDS OF 5 CARDS TO EACH OF 4 PLAYERS FROM THE STANDARD DECK OF 52 CARDS.

HAND1. $C(52,5)$ COMBINATIONS
HAND2 (WHEN HAND1 CHOSEN) $C(47,5)$
HAND3 (H1, H2 CHOSEN) $C(42,5)$
HAND4 (H1, H2, H3 CHOSEN) $C(37,5)$
BY THE PRODUCT RULE: $\# = C(52,5)\cdot C(47,5)\cdot C(42,5)\cdot C(37,5) =$
$= \frac{52!\cdot 47!\cdot 42!\cdot 37!}{5!\cdot 47!\cdot 5!\cdot 42!\cdot 5!\cdot 37!\cdot 5!\cdot 32!} = \frac{52!}{5!\,5!\,5!\,5!\,32!}$

THE SECOND SOLUTION: # = NUMBER OF PERMUTATIONS WITH 5, 5, 5, 5, 32 INDISTINGUISHABLE OBJECTS

INDEED, THE TYPES OF 52 CARDS CORRESPOND TO HANDS 1, 2, 3, 4 (FIVE CARDS OF EACH TYPE) AND THE REMAINING SET (32 CARDS)
$\# = \frac{52!}{5!\cdot 5!\cdot 5!\cdot 5!\cdot 32!}$

Th. THE NUMBER OF WAYS TO DISTRIBUTE n DISTINCT OBJECTS INTO K DISTINCT BOXES $n_1, \ldots n_k$ OBJECTS EACH IS $\frac{n!}{n_1!\cdot n_2!\cdots n_k!}$