LEC.21 10/18/99

PROBABILITY III.

RANDOM VARIABLE IS A FUNCTION FROM S TO $\mathbb{R}$

SAMPLE SPACE REAL NUMBERS

NOTATION $X(t)\big|_{t\in S}$, $X(s)\big|_{s\in S}$

EXAMPLE: A COIN IS FLIPPED n TIMES
$X(t)$ = THE NUMER OF HEADS IN t

$n=3$
$X(TTT)=0$
$X(HTT)=X(THT)=X(TTH)=1$
$X(HHT)=X(HTH)=X(THH)=2$
$X(HHH)=3$

EXAMPLE: $X(t)$ = the SUM OF NUMBERS THAT APPEAR WHEN A PAIR OF DICE IS ROLLED

$X(1,1)=2$
$X(2,1)=X(1,2)=3$
- - - - - -
$X(5,6)=X(6,5)=11$
$X(6,6)=12$

NOTE: RANDOM VARIABLE IS NEITHER VARIABLE NOR RANDOM

THE EXPECTED VALUE (OR EXPECTATION) of X

$$E(X)=\sum_{s\in S} p(s)\cdot X(s)$$

EXAMPLE: $X(s)$ = NUMBER OF HEADS IN s

$n=3$ $E(X)=\frac{1}{8}\cdot 0+\frac{1}{8}\cdot 1+\frac{1}{8}\cdot 1+\frac{1}{8}\cdot 1+\frac{1}{8}\cdot 2+\frac{1}{8}\cdot 2+\frac{1}{8}\cdot 2+\frac{1}{8}\cdot 3$
$=0\cdot\frac{1}{8}+1\cdot\frac{3}{8}+2\cdot\frac{3}{8}+3\cdot\frac{1}{8}=\frac{3}{2}$

A PRACTICAL FORMULA: $E(X)=\sum_{z\in X(S)} p(X=z)\cdot z$

COUNTING $E(X)$ BY LARGER PORTIONS

EXAMPLE: $X(S)$ = THE SUM ON A PAIR OF DICE

$E(X)=2\cdot p(X=2)+3\cdot p(X=3)+4\cdot p(X=4)+\cdots+12\cdot p(X=12)$
$=2\cdot\frac{1}{36}+3\cdot\frac{2}{36}+4\cdot\frac{3}{36}+5\cdot\frac{4}{36}+6\cdot\frac{5}{36}+7\cdot\frac{6}{36}+$
$+8\cdot\frac{5}{36}+9\cdot\frac{4}{36}+10\cdot\frac{3}{36}+11\cdot\frac{2}{36}+12\cdot\frac{1}{36}=7$

EXAMPLE: n BERNOULLI TRIALS

$$E(X)=\sum_{k=1}^{n} k\cdot p(X=k)=\sum_{k=1}^{n} k\,C(n,k)\,p^k q^{n-k}$$

NOTE: $k\cdot C(n,k)=\frac{k\cdot n!}{k!\,(n-k)!}=\frac{n!}{(k-1)!\,(n-k)!}=\frac{n\cdot(n-1)!}{(k-1)!\,(n-k)!}=n\,C(n-1,k-1)$

So, $E(X)=\sum_{k=1}^{n} n\cdot C(n-1,k-1)\,p^k q^{n-k}=$
$=np\cdot\sum_{k=1}^{n} C(n-1,k-1)\,p^{k-1}q^{(n-1)-(k-1)}=$
$=np\cdot\underbrace{\sum_{j=0}^{n-1} C(n-1,j)\cdot p^j\cdot q^{(n-1)-j}}_{\text{BY THE BINOMIAL FORMULA }=(p+q)^{n-1}=1^{n-1}=1}=np$

Th. X,Y ARE RANDOM VARIABLES ON A SPACE S.
$E(X+Y)=E(X)+E(Y)$
$E(aX)=a\cdot E(X)$

PROOF. $E(X+Y)=\sum p(s)\big(X(s)+Y(s)\big)=\sum p(s)X(s)+$
$+\sum p(s)\cdot Y(s)=E(X)+E(Y)$
$E(aX)=\sum p(s)\cdot a\cdot X(s)=a\sum p(s)\cdot X(s)=aE(X)$

COR. $E(X_1+\cdots+X_n)=E(X_1)+\cdots+E(X_n)$
$E(aX+b)=aE(X)+b$

$E(aX+b)=E(aX)+E(b)=aE(X)+E(b)=aE(X)+b$

$E(b)=\sum p(s)\cdot b=b\cdot\sum p(s)=b\cdot 1=b$

INDEPENDENT RANDOM VARIABLES IF

$p(X=a \text{ AND } Y=b)=p(X=a)\cdot p(Y=b)$

i.e. THE EVENTS $X=a$ AND $Y=b$ ARE INDEPENDENT FOR ALL a,b.

EXAMPLE $X_1(s)$ = THE NUMBER ON THE FIRST DIE
$X_2(s)$ = — " — SECOND — " —

$X_1(i,j)=i$ $X_2(i,j)=j$

$\left.\begin{array}{l} p(X_1=i \text{ AND } X_2=j)=p(i,j)=\frac{1}{36} \\ p(X_1=i)=\frac{1}{6}\ ;\ p(X_2=j)=\frac{1}{6}\end{array}\right\}\Rightarrow X_1, X_2$ ARE INDEPENDENT

X_1 AND $X_3=X_1+X_2$ ARE NOT INDEPENDENT

INDEED $p(X_1=1 \text{ AND } X_3=10)=0$
$p(X_1=1)=\frac{1}{6}$; $p(X_3=10)=\frac{3}{36}=\frac{1}{12}\neq 0$

Th. IF X,Y ARE INDEPENDENT THEN $E(XY)=E(X)\cdot E(Y)$

PROOF. $E(XY)=\sum_{\substack{a\in X(S)\\ b\in Y(S)}} a\cdot b\cdot p(X=a \text{ AND } Y=b)=$
$=\sum a\cdot b\cdot p(X=a)\cdot p(Y=b)=$
$=\left[\sum a\cdot p(X=a)\right]\cdot\left[\sum b\cdot p(Y=b)\right]=$
$=E(X)\cdot E(Y)$

THE VARIANCE OF X IS A MEASURE OF HOW WIDELY X IS DISTRIBUTED AROUND ITS EXPECTED VALUE $E(X)$. THE NAIVE TRY: $V = E(X-E(X))$ FAILS SINCE $E(X-E(X)) = E(X)-E(X) = 0$

+'S AND −'S AROUND $E(X)$ ARE BALANCED! $\Leftarrow$

$$V = E(X-E(X))^2 = \sum_{s \in S} [X(s)-E(X)]^2 \cdot p(s)$$

STANDARD DEVIATION $\sigma(X) = \sqrt{V(X)}$

Th. $V = E(X^2) - [E(X)]^2$

PROOF $V = E(X-E(X))^2 = E[X^2 - 2X\,E(X) + [E(X)]^2] =$

$= E(X^2) - E(2X\,E(X)) + [E(X)]^2 =$

$= E(X^2) - 2 \cdot E(X) \cdot E(X) + [E(X)]^2 =$

$= E(X^2) - [E(X)]^2$

EXAMPLE. ONE BERNOULLI TRIAL $X(t) = \begin{cases} 1, & \text{IF } t = \text{SUC} \\ 0, & \text{IF } t = \text{FAIL} \end{cases}$

NOTE, THAT $X^2(t) = X(t)$. $E(X) = p = E(X^2)$

$V(X) = E(X^2) - [E(X)]^2 = p - p^2 = p(1-p) = p \cdot q$

Th. IF $X_1, \dots, X_n$ ARE PAIRWISE INDEPENDENT THEN $V(X_1 + \dots + X_n) = V(X_1) + \dots + V(X_n)$

PROOF. $n=2$ $V(X+Y) = E((X+Y)^2) - [E(X+Y)]^2 =$

$= E(X^2 + 2XY + Y^2) - [E(X) + E(Y)]^2 =$

$= E(X^2) + 2E(XY) + E(Y^2) - [E(X)]^2 - 2E(X) \cdot E(Y) - [E(Y)]^2 =$

$= \underbrace{E(X^2) - [E(X)]^2}_{V(X)} + \underbrace{E(Y^2) - [E(Y)]^2}_{V(Y)} + 2\underbrace{[E(X \cdot Y) - E(X) \cdot E(Y)]}_{0,\ \text{SINCE } E(X \cdot Y) = E(X)E(Y)}$

$= V(X) + V(Y)$

EXAMPLE: TWO DICE $X_1(ij) = i$; $X_2(ij) = j$ INDEPENDENT!

$X = X_1 + X_2$

$V(X) = V(X_1 + X_2) = V(X_1) + V(X_2)$

$V(X_1) = E(X_1^2) - [E(X_1)]^2 = [1^2 + 2^2 + \dots + 6^2] \cdot \frac{1}{6} - [(1+2+\dots+6)\frac{1}{6}]^2 =$

$= \frac{35}{12} = V(X_2)$. THEREFORE $V(X) = \frac{35}{12} + \frac{35}{12} = \frac{35}{6}$

$\sigma(X) = \sqrt{35/6}$

AVERAGE-CASE COMPUTATIONAL COMPLEXITY

$\xrightarrow[s \in S]{\text{INPUT}} \boxed{\mathcal{A}}$ $X(s)$ = THE NUMBER OF OPERATIONS USED BY $\mathcal{A}$ ON AN INPUT s

1. ASSIGN A PROBABILITY $p(s)$ TO EACH POSSIBLE INPUT VALUE s // USUALLY THE MOST CONTROVERSIAL PART!

2. $E(X)$ IS THE AVERAGE-CASE COMPLEXITY

EXAMPLE. THE AVERAGE CASE COMPLEXITY OF THE LINEAR SEARCH ALGORITHM.

n NUMBERS, A SAMPLE NUMBER x

$2i+1$ COMPARISONS IF x is i-th in the list

p = PROBABILITY THAT x IS IN THE LIST

$q = 1-p$ — " — " — NOT — " —

p/n = PROBABILITY THAT x IS i-th IN THE LIST

$E = 3 \cdot p/n + 5 \cdot p/n + \dots + (2n+1) \cdot p/n + (2n+2) \cdot q =$

$= \frac{p}{n}(3 + 5 + \dots + 2n+1) + (2n+2)q = \frac{p}{n}((n+1)^2 - 1) + (2n+2) \cdot q =$

$= p(n+2) + (2n+2) \cdot q$. IF $p=1$, $E = n+2$ (vs. $2n+1$ IN THE WORST-CASE).