HW 4.3: 8 16b,d 24b 32

LEC. 18 10/08/99

PERMUTATIONS AND COMBINATIONS

A PERMUTATION IS AN ORDERED K-TUPLE OF DISTINCT OBJECTS. (K-PERMUTATION)

EXAMPLE $S = \{a, b, c\}$ PERMUTATIONS OF S:
(a,b,c), (a,c,b) (b,a,c), (b,c,a), (c,a,b), (c,b,a)

TH. THE NUMBER OF r-PERMUTATIONS OF A SET WITH n DISTINCT ELEMENTS IS

$$P(n,r) = n \cdot (n-1) \cdot (n-2) \cdots (n-r+1) = \frac{n!}{(n-r)!}$$

PROOF n WAYS TO CHOOSE THE FIRST ELEMENT
n-1 WAYS TO CHOOSE THE SECOND ELEMENT
FROM n-1 ELEMENTS REMAINING
n-2 WAYS TO CHOOSE THE THIRD ELEMENT
. . . . ..
n-r+1 WAYS TO CHOOSE THE rth ELEMENT

BY THE PRODUCT RULE $P(n,r) = n(n-1) \cdot (n-2) \cdots (n-r+1)$

COR. THE TOTAL # OF PEMUTATIONS OF A SET WITH n DISTINCT ELEMENT IS $n!$

EXAMPLE THE # OF WAYS TO AWARD 3 MEDALS TO 10 PLAYERS IS $P(10,3) = 10 \cdot 9 \cdot 8 = 720$

r-COMBINATION OF S IS A SUBSET OF S WITH r ELEMENTS

$S = \{a, b, c\}$. 2-COMBINATIONS OF S ARE

$\{a,b\}$, $\{a,c\}$, $\{b,c\}$ || 2 PERMUTATIONS OF S ARE (ab) (ba) (ac) (ca) (bc) (cb)

th. THE NUMBER OF r-COMBINATIONS OF A SET WITH n ELEMENTS EQUALS

$$C(n,r) = \frac{n!}{r!\,(n-r)!}$$

|| CALLED BINOMIAL COEFFICIENTS ALTERNATIVE NOTATIONS C_n^k, $\binom{k}{n}$

PROOF $\underbrace{P(n,r)}_{\text{CHOOSING } r\text{-PERMUTATION}} = \underbrace{C(n,r)}_{\text{CHOOSING } r\text{-COMBINATION}} \cdot \underbrace{P(r,r)}_{\text{ORDERING THE SUBSET OF } r \text{ ELEMENTS CHOSEN.}}$

CHOOSING r-PERMUTATION = CHOOSING r-COMBINATION AND THEN ORDERING THE SUBSET OF r ELEMENTS CHOSEN.

$$C(n,r) = \frac{P(n,r)}{P(r,r)} = \frac{n!}{(n-r)!\,r!}$$

EXAMPLE THE # OF WAYS TO PICK A TEAM OF THREE HACKERS OUT OF 20 TO REPRESENT CORNELL IN A COMPETITION IN EUROPE IS $C(20,3) = \frac{20!}{3!\,(20-3)!} = \frac{20!}{3!\,17!} =$

$$= \frac{20\cdot 19\cdot 18\cdot \cancel{17!}}{3!\cdot \cancel{17!}} = \frac{20\cdot 19\cdot 18}{6} = 20\cdot 19\cdot 3 =$$

$$= 60\cdot 19 = 1140$$

Th. (PASCAL'S IDENTITY)

$$C(n+1, k) = C(n, k-1) + C(n, k)$$

<u>PROOF.</u> $|T| = n+1$, $a \in T$, $S := T - \{a\}$

ANY K-SUBSET X OF T

1° $a \in X$ THEN $X' = X - \{a\}$ IS A $(k-1)$-SUBSET OF S $C(n, k-1)$ COMBINATIONS

OR 2° $a \notin X$ THE X IS A k-SUBSET OF S $C(n, k)$ COMBINATIONS

BY THE SUM RULE THE TOTAL IS $C(n, k-1) + C(n, k)$

PASCAL TRIANGLE

	$n =$
1	0
1 1	1
1 2 1	2
1 3 3 1	3
1 4 6 4 1	4
1 5 10 10 5 1	5
1 6 15 20 15 6 1	6
1 7 21 35 35 21 7 1	7
1 8 28 56 70 56 28 8 1	8

THE ENTRIES OF THE n-th ROW $= C_n^0 \; C_n^1 \; C_n^2 \dots C_n^{n-1} \; C_n^n$

$$\begin{array}{ccccc} C_n^0 & C_n^1 & C_n^2 & \dots & C_n^{n-1} & C_n^n \\ \| & \| & \| & & \| & \| \\ 1 & n & \frac{n(n-1)}{2} & \dots & n & 1 \end{array}$$

<u>PROOF</u> BY PASCAL'S IDENTITY, SINCE EACH ENTRY IS THE SUM OF TWO ADJACENT ENTRIES ABOVE.

$$\text{th.}\quad \sum_{k=0}^{n} C_n^k = 2^n$$

PROOF: $|\mathcal{P}(S)| = 2^{|S|}$

$2^n = $

TOTAL # OF SUBSETS OF S =

$= C_n^0 + C_n^1 + C_n^2 + \dots + C_n^{n-1} + C_n^n$

th. THE BINOMIAL THEOREM

$$(x+y)^n = C_n^0 \cdot x^n + C_n^1 \cdot x^{n-1} y + C_n^2 x^{n-2} \cdot y^2 + \dots C_n^{n-1} x y^{n-1} + C_n^n y^n$$

$$= x^n + n \cdot x^{n-1} \cdot y + \frac{n(n-1)}{2} x^{n-2} \cdot y^2 + \dots n x y^{n-1} + y^n$$

PROOF $(x+y)^n = \underbrace{\overset{\#1}{(x+y)} \cdot \overset{\#2}{(x+y)} \cdot \overset{\#3}{(x+y)} \cdot \;\dots\; \overset{\#n}{(x+y)}}_{n \text{ SUMS}}$

AFTER EXPANSION WE GET C_n^{n-j} TERMS $x^{n-j} \cdot y^j$
INDEED, TO OBTAIN SUCH A TERM ONE HAS TO CHOOSE $n-j$ x-s FROM #1, #2, …, #n (THE REST #'S GIVE y'S)
THERE ARE C_n^{n-j} WAYS TO MAKE SUCH CHOICE $\Rightarrow$
THERE ARE C_n^{n-j} TERMS $x^{n-j} y^j$ IN THE EXPANDED FORM OF $(x+y)^n$.

EXAMPLE: $(x+y)^3 = (x+y)(x+y)(x+y) = \underbrace{xxx}_{x^3} + \underbrace{xxy + xyx + yxx}_{x^2y} + \underbrace{xyy + yxy + yyx}_{xy^2} + \underbrace{yyy}_{y} = x^3 + 3x^2y + 3xy^2 + y^3$

$$\text{th.}\quad C_n^0 - C_n^1 + C_n^2 - C_n^3 + \dots + (-1)^{n-1} \cdot C_n^{n-1} + (-1)^n C_n^n = 0 \qquad \sum_{k=0}^{n} (-1)^k C_n^k$$

PROOF $0 = ((-1)+1)^n = \sum_{k=0}^{n} C_n^k (-1)^k \cdot 1^{n-k} = \sum_{k=0}^{n} C_n^k (-1)^k$