

LEC 17 10/04/99

COUNTING

EXAMPLE. THERE ARE THREE REPUBLICANS AND TWO DEMOCRATS RUNNING FOR PREZ NOMINATIONS. HOW MANY POSSIBILITIES FOR THE NEXT PRESIDENT ARE THERE?

SOLUTION: THREE VARIANTS FROM G.O.P.
TWO VARIANTS FROM DEM
 $3 + 2 = 5$ ALL TOGETHER

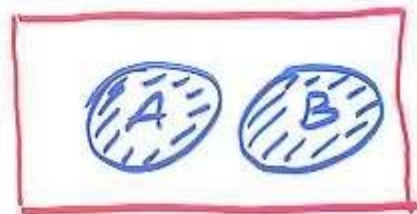
TH. (THE SUM RULE) IF EACH SOLUTION OF A TASK T IS EITHER A SOLUTION OF T_1 OR A SOLUTION OF T_2 AND $SOL T_1 \cap SOL T_2 = \emptyset$ THEN

$$|SOL T| = |SOL T_1| + |SOL T_2|$$

PROOF. $|A \cup B| = |A| + |B|$, IF $A \cap B = \emptyset$



$$A \cap B = \emptyset$$



$$A \cup B$$

EXAMPLE

AN ASSISTANT PROFESSOR IS LOOKING FOR A COMPUTER FOR HER OFFICE. THERE ARE 13 POWER PCs, 22 POWER MACS AND 7 SUNS IN A STORE. HOW MANY CHOICES DOES SHE HAVE?

$$13 + 22 + 7 = 42$$

NOTE THAT WE HAVE USED THE SUM RULE TWICE.

EXAMPLE. AFTER THE FOLLOWING CODE HAS BEEN EXECUTED $k = ?$

$k := 0$

FOR $i_1 := 1$ TO n_1 } T_1
 $k := k + 1$

FOR $i_2 := 1$ TO n_2 } T_2
 $k := k + 1$

...

FOR $i_m := 1$ TO n_m } T_m
 $k := k + 1$

THE TASK OF COMPUTING k IS SPLIT INTO $T_1, T_2, \dots, T_m$ EACH HAVING $n_1, n_2, \dots, n_m$ SOLUTIONS. BY THE SUM RULE
 $k = n_1 + n_2 + \dots + n_m$

EXAMPLE, THERE ARE THREE GOP AND TWO DEM
PRESIDENTIAL CANDIDATES. HOW MANY
POSSIBILITIES TO HAVE A CONTESTING PAIR
ON THE ELECTION DAY IN 2000 DO WE HAVE?

G.O.P. = {B, D, C} DEM = {G, B}

PAIRS: {BG} {BB} {DG} {DB} {CG} {CB}

THREE CHOICES FOR A BLUE POSITION

TWO CHOICES FOR A GREEN POSITION

ALLTOGETHER $3 \cdot 2 = 6$ VARIANTS

TH. (THE PRODUCT RULE) IF EACH SOLUTION IS A PAIR
AND WE HAVE n_1 AND n_2 CHOICES TO FILL
THE COORDINATES THEN THERE ARE $n_1 \cdot n_2$ SOLUTIONS.

PROOF $|A \times B| = |A| \cdot |B|$

EXAMPLE CHAIR IN A THEATER ARE LABELED (L, N)
WHERE L IS A LETTER AND N A POSITIVE
INTEGER ≤ 100 . HOW MANY LABELS ARE THERE?

$26 \cdot 100 = 2600$, BY THE PRODUCT RULE.

EXAMPLE, THE NUMBER OF ALL BIT STRINGS OF LENGTH
TEN?

10 POSITIONS, 2 CHOICES FOR EACH. BY THE
MULTIPLE PRODUCT RULE THE NUMBER OF STRINGS
IS $2 \cdot 2 \cdot 2 \cdot 2 \cdot 2 \cdot 2 \cdot 2 \cdot 2 \cdot 2 \cdot 2 = 2^{10} = 1024$

HOW MANY FUNCTIONS ARE THERE FROM A SET OF m ELEMENTS TO A SET OF n ELEMENTS?

Th. IF $|A|=m$, $|B|=n$ THEN THERE ARE n^m FUNCTIONS FROM A TO B .

PROOF. EACH FUNCTION CAN BE REPRESENTED BY A STRING OF ITS VALUES AT $\{a_1, \dots, a_m\} = A$

$$A = \{x, y, z\} \quad f: (a_1, a_2, a_3) = (f(x), f(y), f(z))$$

$\uparrow \quad \uparrow \quad \uparrow$
EACH IS 0 OR 1

THERE ARE $|A|=m$ POSITIONS TO FILL
AND $|B|=n$ CHOICES AT EACH POSITION
BY THE MULTIPLE PRODUCT RULE THE TOTAL
NUMBER OF SUCH STRINGS IS

$$\underbrace{n \cdot n \cdot n \cdots n}_{m \text{ TIMES}} = n^m$$

COUNTING ONE-TO-ONE FUNCTIONS FROM A TO B

$|A|=m$, $|B|=n$. IF $m > n$ THERE ARE NONE.
ASSUME $m \leq n$. FUNCTION = STRING OF LENGTH m
FIRST POSITION n CHOICES
SECOND $n-1$
...
MTH POSITION $n-m+1$ CHOICES

ANSWER
 $n \cdot (n-1) \cdot (n-2) \cdots (n-m+1)$

MORE COMPLEX PROBLEMS

EXAMPLE A PASSWORD IS SIX TO EIGHT CHARACTERS LONG. EACH CHARACTER IS A DIGIT, A LOWER CASE LETTER OR AN UPPER CASE LETTER. THERE SHOULD BE AT LEAST ONE DIGIT. THE NUMBER OF PASSWORDS?

$$P = P_6 + P_7 + P_8$$

↑ ↑ ↑
POSSIBLE LENGTH.

$$P_6 = |\text{WORDS WITH LETTERS AND DIGITS}| - |\text{WORDS WITHOUT DIGITS}| =$$
$$= 62^6 - 52^6$$
$$P_7 = 62^7 - 52^7$$
$$P_8 = 62^8 - 52^8$$

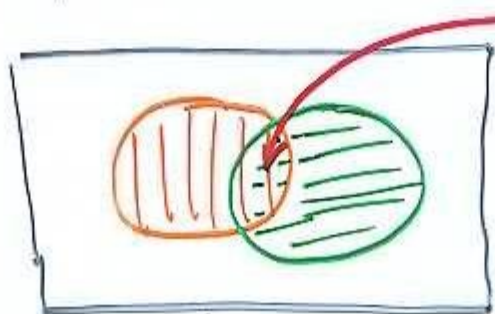
26 LOWER CASE LETTERS
26 UPPER CASE LETTERS
10 DIGITS

$$P = P_6 + P_7 + P_8 \quad \text{A HUGE NUMBER!}$$

TH. (THE INCLUSION-EXCLUSION PRINCIPLE)

$$|A \cup B| = |A| + |B| - |A \cap B|$$

PROOF



HAVE BEEN
COUNTED TWICE!
(IN $|A| + |B|$).

EXAMPLE. AN EDUCATIONAL COMMITTEE HAS SEVEN ELECTED POLITICIANS AND TEN TEACHERS. HOW MANY PERSONS THERE ARE EITHER POLITICIANS OR TEACHERS ? TWO OF THE POLITICIANS ARE

$$|P \cup T| = |P| + |T| - |P \cap T| = \text{TEACHERS}$$

$$= 7 + 10 - 2 = 15$$

TH. (THE PIGEONHOLE PRINCIPLE) IF $\geq k+1$ OBJECTS ARE PLACED INTO k BOXES THEN THERE IS AT LEAST ONE BOX CONTAINING TWO OR MORE OBJECTS

EXAMPLE ANY GROUP OF 367 PEOPLE HAS A PAIR WITH THE SAME BIRTHDAY

TH. (THE GENERALIZED PIGEONHOLE) N OBJECTS, k BOXES $\Rightarrow$ THERE IS A BOX CONTAINING AT LEAST $\lfloor N/k \rfloor$ OBJECTS.

PROOF. OW $k \cdot (\lfloor N/k \rfloor - 1) < k \cdot ((N/k) + 1 - 1) = N$
THE TOTAL # OF OBJECTS $\uparrow$

EXAMPLE. AMONG 100 PEOPLE THERE ARE AT LEAST $\lfloor 100/12 \rfloor = 8$ WHO WERE BORN IN THE SAME MONTH.

HW 4.1: 18a, f, h 24 386 46
4.2: 49, 20, 34