LEC 17 10/04/99

COUNTING

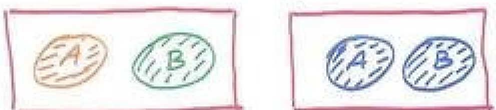
EXAMPLE. THERE ARE THREE REPUBLICANS AND TWO DEMOCRATS RUNNING FOR PREZ NOMINATIONS. HOW MANY POSSIBILITIES FOR THE NEXT PRESIDENT ARE THERE?

SOLUTION: THREE VARIANTS FROM G.O.P.
TWO VARIANTS FROM DEM
$3+2=5$ ALL TOGETHER

Th. (THE SUM RULE) IF EACH SOLUTION OF A TASK T IS EITHER A SOLUTION OF T_1 OR A SOLUTION OF T_2 AND $\mathrm{Sol}\,T_1 \cap \mathrm{Sol}\,T_2 = \emptyset$ THEN

$$|\mathrm{Sol}\,T| = |\mathrm{Sol}\,T_1| + |\mathrm{Sol}\,T_2|$$

PROOF. $|A \cup B| = |A| + |B|$, IF $A \cap B = \emptyset$



$A \cap B = \emptyset$ $\qquad$ $A \cup B$

EXAMPLE. AN ASSISTANT PROFESSOR IS LOOKING FOR A COMPUTER FOR HER OFFICE. THERE ARE 13 POWER PC S, 22 POWER MACS AND 7 SUNS IN A STORE. HOW MANY CHOICES DOES SHE HAVE?

$$13+22+7=42$$

NOTE THAT WE HAVE USED THE SUM RULE TWICE.

EXAMPLE. AFTER THE FOLLOWING CODE HAS BEEN EXECUTED $k=?$

```
k:=0
FOR i1:=1 TO n1     } T1
    k:=k+1
FOR i2:=1 TO n2     } T2
    k:=k+1
. . . . . . . .
FOR im:=1 TO nm     } Tm
    k:=k+1
```

THE TASK OF COMPUTING k IS SPLIT INTO $T_1, T_2, \ldots, T_m$ EACH HAVING $n_1, n_2, \ldots n_m$ SOLUTIONS. BY THE SUM RULE

$$k = n_1 + n_2 + \cdots + n_m$$

EXAMPLE. THERE ARE THREE G.O.P. AND TWO DEM PRESIDENTIAL CANDIDATES. HOW MANY POSSIBILITIES TO HAVE A CONTESTING PAIR ON THE ELECTION DAY IN 2000 DO WE HAVE?

$G.O.P. = \{B, D, C\}$ $\qquad$ $DEM = \{G, B\}$

PAIRS: {BG} {BB} {DG} {DB} {CG} {CB}

THREE CHOICES FOR A BLUE POSITION
TWO CHOICES FOR A GREEN POSITION
ALLTOGETHER $3 \cdot 2 = 6$ VARIANTS

Th. (THE PRODUCT RULE) IF EACH SOLUTION IS A PAIR AND WE HAVE n_1 AND n_2 CHOICES TO FILL THE COORDINATES THEN THERE ARE $n_1 \cdot n_2$ SOLUTIONS.

PROOF $|A \times B| = |A| \cdot |B|$

EXAMPLE CHAIR IN A THEATER ARE LABELED (L,N) WHERE L IS A LETTER AND N A POSITIVE INTEGER ≤ 100. HOW MANY LABELS ARE THERE?

$26 \cdot 100 = 2600$, BY THE PRODUCT RULE.

EXAMPLE. THE NUMBER OF ALL BIT STRINGS OF LENGTH TEN?

10 POSITIONS, 2 CHOICES FOR EACH. BY THE MULTIPLE PRODUCT RULE THE NUMBER OF STRINGS IS $2 \cdot 2 \cdot 2 \cdot 2 \cdot 2 \cdot 2 \cdot 2 \cdot 2 \cdot 2 \cdot 2 = 2^{10} = 1024$

HOW MANY FUNCTIONS ARE THERE FROM A SET OF m ELEMENTS TO A SET OF n ELEMENTS?

Th. IF $|A| = m$, $|B| = n$ THEN THERE ARE n^m FUNCTIONS FROM A TO B.

PROOF. EACH FUNCTION CAN BE REPRESENTED BY A STRING OF ITS VALUES AT $\{a_1, \ldots, a_n\} = A$

$A = \{x\ y\ z\}$ $\qquad$ $f: (6, 6_2 6_3) = (f(x), f(y), f(z))$

$B = \{0\,1\}$ $\qquad$ EACH IS 0 OR 1

THERE ARE $|A| = m$ POSITIONS TO FILL AND $|B| = n$ CHOICES AT EACH POSITION BY THE MULTIPLE PRODUCT RULE THE TOTAL NUMBER OF SUCH STRINGS IS

$$\underbrace{n \cdot n \cdot n \cdots n}_{m \text{ TIMES}} = n^m$$

COUNTING ONE-TO-ONE FUNCTIONS FROM A TO B
$|A| = m$, $|B| = n$. IF $m > n$ THERE ARE NONE.
ASSUME $m \le n$. FUNCTION = STRING OF LENGTH m

FIRST POSITION	n CHOICES	ANSWER
SECOND	$n-1$	$n \cdot (n-1) \cdot (n-2) \cdot \ldots \cdot (n-m+1)$.
mth POSITION	$n-m+1$ CHOICES	

MORE COMPLEX PROBLEMS

<u>EXAMPLE</u> A PASSWORD IS SIX TO EIGHT CHARACTERS LONG. EACH CHARACTER IS A DIGIT, A LOWER CASE LETTER OR AN UPPER CASE LETTER. THERE SHOULD BE AT LEAST ONE DIGIT. THE NUMBER OF PASSWORDS?

$P = P_6 + P_7 + P_8$

POSSIBLE LENGTH.

P_6 = |WORDS WITH LETTERS AND DIGITS| − |WORDS WITHOUT DIGITS| =

$= 62^6 - 52^6$

26 LOWER CASE LETTERS
26 UPPER CASE LETTERS
10 DIGITS

$P_7 = 62^7 - 52^7$

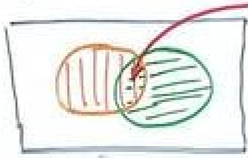
$P_8 = 62^8 - 52^8$

$P = P_6 + P_7 + P_8$ A HUGE NUMBER!

Th. (THE INCLUSION-EXCLUSION PRINCIPLE)

$$|A \cup B| = |A| + |B| - |A \cap B|$$

<u>PROOF</u>



HAVE BEEN COUNTED TWICE! (IN $|A|+|B|$)

<u>EXAMPLE.</u> AN EDUCATIONAL COMMITTEE HAS SEVEN ELECTED POLITICIANS AND TEN TEACHERS. HOW MANY PERSONS THERE ARE EITHER POLITICIANS OR TEACHERS? TWO OF THE POLITICIANS ARE TEACHERS

$$|P \cup T| = |P| + |T| - |P \cap T| =$$
$$= 7 + 10 - 2 = 15$$

Th. (THE PIGEONHOLE PRINCIPLE) IF $\geq k+1$ OBJECTS ARE PLACED INTO k BOXES THEN THERE IS AT LEAST ONE BOX CONTAINING TWO OR MORE OBJECTS

<u>EXAMPLE</u> ANY GROUP OF 367 PEOPLE HAS A PAIR WITH THE SAME BIRTHDAY

Th. (THE GENERALIZED PIGEONHOLE) N OBJECTS, k BOXES $\Rightarrow$ THERE IS A BOX CONTAINING AT LEAST $\lceil N/k \rceil$ OBJECTS.

<u>PROOF</u>. OW $k \cdot (\lceil N/k \rceil - 1) < k\left((N/k) + 1 - 1\right) = N$
THE TOTAL # OF OBJECTS ↑

<u>EXAMPLE</u>. AMONG 100 PEOPLE THERE ARE AT LEAST $\lceil 100/12 \rceil = 9$ WHO WERE BORN IN THE SAME MONTH.

HW 4.1: 18a,f,h 24 38b 46
4.2: 4g, 20, 34