LEC. 14 9/27/99

RECURSIVE DEFINITIONS

1. $f(0) = a$
2. $f(n+1) =$ SOME FUNCTION OF $f(0), f(1), \dots, f(n)$

EXAMPLES 1. $\begin{cases} f(0) = 0 \\ f(n+1) = f(n) + (n+1) \end{cases}$

$f(0) = 0;\ f(1) = 0+1,\ f(2) = 0+1+2, \dots, f(n) = 1+2+\dots+n$

2. $\begin{cases} p(0) = 1 \\ p(n+1) = p(n)\cdot(n+1) \end{cases}$ $\quad p(1) = 1,\ p(2) = 1\cdot 2,\ p(3) = 1\cdot 2\cdot 3, \dots$ $\quad p(n) = 1\cdot 2\cdot 3 \cdot \cdots \cdot n = n!$

3. $\begin{cases} a(0) = a_0 \\ a(n+1) = a(n) + d \end{cases}$ $\quad a(0) = a_0,\ a(1) = a_0 + d,\ a(2) = a_0 + 2d, \dots$ $\quad \boxed{a_n = a_0 + n\cdot d}$

↑ RECURSIVE DEFINITION OF ARITHMETICAL PROGRESSION

4. $\begin{cases} f(0) = 1 \\ f(n+1) = f(n)\cdot a \end{cases}$ $\quad f(0) = 1,\ f(1) = a,\ f(2) = a\cdot a, \dots$ $\quad f(n) = a^n$

↑ RECURSIVE DEFINITION OF x^n FOR NONNEGATIVE INTEGERS n

5. $$\begin{cases} x\cdot 0 = 0 \\ x\cdot(y+1) = x\cdot y + x \end{cases}$$

RECURSIVE DEFINITION OF MULTIPLICATION VIA ADDITION

$x\cdot 0 = 0$; $x\cdot 1 = x\cdot(0+1) = x\cdot 0 + x = x$

$x\cdot 2 = x\cdot(1+1) = x\cdot 1 + x = x+x$

$x\cdot 3 = x\cdot(2+1) = x\cdot 2 + x = (x+x)+x$, etc.

6. $a_0, a_1, a_2, \ldots, a_n, \ldots$ A SEQUENCE

$$\begin{cases} S(0) = 0 \\ S(n+1) = S(n) + a_{n+1} \end{cases} \qquad \begin{cases} P(0) = 1 \\ P(n+1) = P(n)\cdot a_{n+1} \end{cases}$$

$$S(n) = \sum_{i=0}^{n} a_i \qquad P(n) = \prod_{i=0}^{n} a_i$$

↑ RECURSIVE DEFINITIONS OF <u>SUM</u> AND <u>PRODUCT</u> OF A SEQUENCE

7. <u>FIBONACCI NUMBERS</u>

$$\begin{cases} f_0 = 0,\ f_1 = 1 \\ f_n = f_{n-1} + f_{n-2} \end{cases}$$

$0, 1, 1, 2, 3, 5, 8, 13, 21, \ldots$

$$\begin{cases} f(0) = 0 \\ f(n+1) = \begin{cases} 1, & \text{IF } n = 0 \\ f(n) + f(n-1) & \text{O.W.} \end{cases} \end{cases}$$

A MORE STANDARD RECURSIVE REPRESENTATION

Th. LET $\alpha := \frac{1+\sqrt{5}}{2}$. THEN $f_n > \alpha^{n-2}$ WHENEVER $n \geq 3$.

PROOF. BY THE SECOND PRINCIPLE OF MATHEMATICAL INDUCTION

BASE: $n=3, n=4$. $f_3 = 2$.

$\sqrt{5} < 3$

$1+\sqrt{5} < 1+3 = 4$

$\frac{1+\sqrt{5}}{2} < \frac{4}{2} = 2 = f_3$

$f_4 = 3$. $\alpha^2 = \frac{1}{4}(1+\sqrt{5})^2 =$

$= \frac{1}{4}(1+2\sqrt{5}+5) = \frac{1}{4}(6+2\sqrt{5}) =$

$= \frac{3+\sqrt{5}}{2} < \frac{3+3}{2} = 3 = f_4$

STEP. INDUCTION HYPOTHESIS: $f_3 > \alpha, f_4 > \alpha^2, \dots, f_n > \alpha^{n-2}$

$f_{n+1} = f_n + f_{n-1}$

$$+\begin{cases} f_n > \alpha^{n-2} \\ f_{n-1} > \alpha^{n-3} \end{cases}$$

NOTE: $\alpha^2 = \frac{3+\sqrt{5}}{2} = \alpha + 1$

$f_n + f_{n-1} > \alpha^{n-2} + \alpha^{n-3} = \alpha^{n-3}(\alpha+1) =$

$= \alpha^{n-3} \cdot \alpha^2 = \alpha^{n-1}$

THEREFORE $f_{n+1} > \alpha^{n-1}$

th. LAMÉ'S THEOREM. LET $a \geq b > 0$ BE INTEGERS. THEN THE NUMBER OF DIVISIONS USED BY THE EUCLIDEAN ALGORITHM TO FIND $\gcd(a,b)$ 5 × THE NUMBER OF DECIMAL DIGITS IN b.

PROOF.

$$a = b q_1 + r_2 \qquad r_2 < b \qquad b \geq r_2 + r_3 \geq f_n + f_{n-1} = f_{n+1}$$

$$b = r_2 \cdot q_2 + r_3 \qquad r_3 < r_2 \qquad r_2 \geq r_3 + r_4 \geq f_{n-1} + f_{n-2} = f_n$$

$$\cdots \qquad \cdots \qquad \cdots$$

$$r_{n-2} = r_{n-1} q_{n-1} + r_n \qquad r_n < r_{n-1} \qquad r_{n-2} \geq r_{n-1} + r_n \geq \overbrace{f_3 + f_2}^{f_4}$$

$$\underbrace{r_{n-1} = r_n \cdot q_n} \quad \Longrightarrow \quad r_{n-1} \geq 2 \cdot r_n \geq 2 f_2 = f_3$$

$$r_n \geq 1 = f_2$$

$q_n \geq 2$, SINCE $r_n < r_{n-1}$

$$r_n = \gcd(a,b)$$

$$b \geq f_{n+1} > \alpha^{n-1} \qquad \log_{10} \alpha \sim 0.208 > 1/5$$

$$\log_{10} b > (n-1) \cdot \log_{10} \alpha > \frac{n-1}{5}$$

$n - 1 < 5 \cdot \log_{10} b$; IF $b < 10^k$ (k DIGITS)

$n - 1 < 5 \cdot k$; $\boxed{n \leq 5 \cdot k}$

RECURSIVELY DEFINED SETS

EXAMPLE 1. S IS DEFINED BY

A. $2 \in S$ — INITIAL COLLECTION

B. $x \in S \wedge y \in S \rightarrow x+y \in S$ — GENERATING RULES

DEFAULT: THERE IS NOTHING ELSE IN S

S = SET OF ALL EVEN POSITIVE INTEGERS

INDUCTION ON A RECURSIVE DEFINITION OF A SET.

A. THE INITIAL ELEMENTS SATISFY P

B. GENERATING RULES PRESERVE P

THEN ALL ELEMENTS OF S SATISFY P.

ALL ELEMENTS OF S FROM EXAMPLE 1 ARE EVEN POSITIVE INTEGERS. INDEED, A) 2 IS POSITIVE EVEN B) IF x, y ARE POSITIVE EVEN, THEN $x+y$ ALSO IS.

EXAMPLE 2. WELL-FORMED PROPOSITIONS

A. $T, F, \underbrace{p_1, p_2, \dots, p_n, \dots}_{\text{PROPOSITIONAL VARIABLES}}$ ARE W-FP

B. IF G, H ARE W-FP THEN $(\neg G)$, $(G \wedge H)$, $(G \vee H)$, $(G \rightarrow H)$, $(G \leftrightarrow H)$ ARE ALSO W-FP.

$((\neg(p_1 \rightarrow p_2)) \vee (p_2 \leftrightarrow \neg p_3))$, $(\neg(\neg p_i))$, $((T \rightarrow p_1) \leftrightarrow p_1)$

Th. EACH W-FP HAS EQUAL NUMBERS OF (,)

INDUCTION ON THE DEFINITION OF W-FP.

BASIS: T, F, p_i HAVE NO (,) AT ALL

INDUCTIVE STEP. HYPOTHESIS: G, H HAVE ZERO BALANCE OF (,). THEN $(\neg G)$, $(G \wedge H)$, $(G \vee H)$, $(G \to H)$, $(G \leftrightarrow H)$ ALSO HAVE ZERO BALANCE OF (,).

COROLLARY NO PROPER INITIAL PART OF A W-FP IS ITSELF A W-FP

INDEED, EVERY W-FP IS EITHER ATOMIC OR ENDS WITH ")". IN THE FIRST CASE THERE ARE NO NONTRIVIAL INITIAL PARTS AT ALL. IN THE SECOND CASE THE BALANCE OF (,) IN EVERY INITIAL PART IS POSITIVE.

SET OF STRINGS OVER THE ALPHABET Σ.

A. $\lambda \in \Sigma^*$ (λ IS THE EMPTY STRING)

B. $w \in \Sigma^*$, $x \in \Sigma \Rightarrow wx \in \Sigma^*$

LENGTH $\ell(w)$ OF THE STRING w:

A. $\ell(\lambda) = 0$; B. $\ell(wx) = \ell(w) + 1$

Th. $\ell(uy) = \ell(u) + \ell(y)$ FOR ALL $y \in \Sigma^*$

INDUCTION ON THE RECURSIVE DEFINITION OF Σ^*

BASIS $y = \lambda$, $\ell(u\lambda) = \ell(u) = \ell(u) + 0 = \ell(u) + \ell(\lambda)$

STEP. $\underbrace{\ell(uy) = \ell(u) + \ell(y)}_{\text{IND. HYPOTHESIS}}$, $a \in \Sigma$. $\ell(uya) = \ell(uy) + 1 =$

$= \ell(u) + \ell(y) + 1 = \ell(u) + \ell(ya)$

HW 3.3: 6b,d 16 32