LEC. 13 09/29/99

INDUCTION

EXAMPLE. HOUSES TO BE BUILT:  1 2 ... n ...

LOCAL REGULATIONS:  $n \to n+1$ $\forall n (R(n) \to R(n+1))$

THE FIRST HOUSE:  $R(1)$

CONCLUSION:  $\forall n\, R(n)$

MATHEMATICAL INDUCTION - A METHOD OF PROVING UNIVERSAL STATEMENTS $\forall n\, P(n)$ FOR POSITIVE (NON NEGATIVE) INTEGERS

$P(n)$ - SOME KIND OF PROPERTY WHICH AN INTEGER n MAY HAVE. THEN $P(n)$ HOLDS FOR ALL n IF 1. AND 2.

e.g. $P(n)$ IS "n^3-n IS DIVISIBLE BY 3" $n^3-n \vdots 3$ (NOTATION)

1. BASIS STEP: ESTABLISH $P(1)$

$P(1)$ IS "1^3-1 IS DIVISIBLE BY 3" TRUE, SINCE $1^3-1=0$ AND $3 \mid 0$

2. INDUCTION STEP: ESTABLISH $P(n) \to P(n+1)$ FOR EACH POSITIVE INTEGER n

SUPPOSE $P(n)$, i.e. $n^3-n \vdots 3$ AND CONSIDER $P(n+1)$

$(n+1)^3-(n+1) = n^3+3n^2+3n+1-n-1 = n^3-n+3(n^2+n) \vdots 3$

SINCE $n^3-n \vdots 3$, BY THE ASSUMPTION.

EXAMPLE. $P(n)$: $1+2+\cdots+n = \frac{n(n+1)}{2}$

1. BASIS: $P(1)$ $1 = \frac{1\cdot(1+1)}{2} = \frac{1\cdot 2}{2} = 1$

2. STEP: $P(n) \overset{?}{\Rightarrow} P(n+1)$. ASSUME $P(n)$ — THE INDUCTION HYPOTHESIS

$P(n+1)$: $\underline{1+2+\cdots+n}+(n+1) = \frac{n(n+1)}{2}+(n+1) =$

$= \frac{n(n+1)}{2}$, BY THE INDUCTION HYPOTHESIS

$= \frac{n(n+1)+2(n+1)}{2} = \frac{(n+1)(n+2)}{2}$

$1+2+\cdots+n = \frac{n(n+1)}{2} \Rightarrow 1+2+\cdots+(n+1) = \frac{(n+1)(n+2)}{2}$!

CONCLUSION: $\forall n\, P(n)$ i.e. $1+2+\cdots+n = \frac{n(n+1)}{2}$ HOLDS FOR ALL $n \geq 1$

EXAMPLE. $P(n)$: $n < 2^n$

1. BASIS, $P(1)$ $1 < 2^1 = 2$

2. INDUCTION STEP: $n < 2^n \overset{?}{\Rightarrow} (n+1) < 2^{(n+1)}$

HYPOTHESIS: $n < 2^n$. THEN $n+1 < 2^n+1 < 2^n+2^n = 2^{n+1}$

CONCLUSION: $n < 2^n$ FOR ALL $n \geq 1$

EXAMPLE SUM OF GEOMETRIC PROGRESSIONS

$P(n)$: $1+q+q^2+\cdots+q^{n-1} = \frac{q^n-1}{q-1}$ $(q \neq 1)$

BASIS. $P(1)$: $1 = \frac{q^1-1}{q-1} = 1$

INDUCTIVE STEP: $P(n) \overset{?}{\Rightarrow} P(n+1)$

HYPOTHESIS: $P(n)$: $1+q+q^2+\cdots+q^{n-1} = \frac{q^n-1}{q-1}$

THEN $\underbrace{1+q+q^2+\cdots+q^{n-1}+q^n}_{P(n+1)} = \frac{q^n-1}{q-1}+q^n = \frac{\cancel{q^n}-1+q^{n+1}-\cancel{q^n}}{q-1}$

$= \frac{q^{n+1}-1}{q-1}$

CONCLUSION: $\forall n\, P(n)$: $1+q+q^2+\cdots+q^{n-1} = \frac{q^n-1}{q-1}$ FOR ALL $n \geq 1$

EXAMPLE NUMBER OF SUBSETS OF A FINITE SET $|X|=n$

$P(n)$: $S(n) = 2^n (= 2^{|X|})$ $S(n)$

BASIS: $P(1)$ $S(1) = 2$ SINCE $\{a\}$ HAS ONLY TWO SUBSETS $\emptyset, \{a\}$

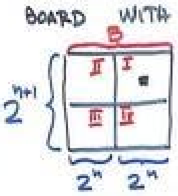
INDUCTIVE STEP: $P(n) \overset{?}{\Rightarrow} P(n+1)$. LET Y BE A SET $|X|=n$; $Y = X \cup \{a\}$, $X = Y - \{a\}$. THERE ARE TWO KINDS OF SUBSETS IN Y:

1. U, $(U \subseteq X)$ 2^n DIFFERENT U's $\Rightarrow 2\cdot 2^n$ SUBSETS $= 2^{n+1}$
2. $U \cup \{a\}$ BY THE ASSUMPTIONS IF Y

EXAMPLE $P(n)$ $2^n \times 2^n$ CHESSBOARD WITH ONE SQUARE REMOVED CAN BE TILED USING L-SHAPED PIECES

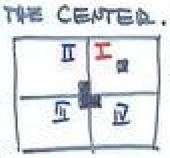
BASIS. $P(1)$: 2×2 CHESS BOARD. ALL POSSIBLE VARIANTS WITH A SQUARE REMOVED CAN BE COVERED AS ABOVE.

INDUCTIVE STEP. ASSUME $P(n)$ i.e. EVERY $2^n \times 2^n$ BOARD MINUS ONE SQUARE CAN BE COVERED. IN ORDER TO ESTABLISH $P(n+1)$ CONSIDER B - a $2^{n+1} \times 2^{n+1}$ BOARD WITH ONE SQUARE REMOVED.



BY THE INDUCTION HYPOTHESIS I CAN BE COVERED.

CUT THREE SQUARES FROM II, III, IV AROUND THE CENTER. THEY CAN BE COVERED BY ONE TILE.



BY THE INDUCTION HYPOTHESIS EACH OF II, III, IV CAN BE COVERED. THEREFORE THE WHOLE OF B CAN BE COVERED.

"THE SECOND PRINCIPLE" OF MATHEMATICAL INDUCTION

BASIS STEP: $P(1)$

INDUCTIVE STEP: $\underbrace{P(1) \wedge P(2) \wedge \dots \wedge P(n)}_{}$ $\longrightarrow P(n+1)$

CONCLUSION: $\forall_n P(n)$

EXTENDED INDUCTIVE HYPOTHESIS

<u>EXAMPLE</u> $P(n)$: "IF $n>1$ THEN n IS A PRODUCT OF PRIMES"

BASIS: $P(1)$ $1>1 \Rightarrow \dots$ TRUE SINCE THE PREMISE OF "$\Rightarrow$" IS FALSE

STEP: ASSUME $P(n)$, AND CONSIDER $n+1$.

1°. $n+1$ IS A PRIME — DONE

2°. $n+1$ IS COMPOSITE $\Rightarrow$ $n+1 = m \cdot k$ SUCH THAT $1 < m, k < n+1$

BY THE INDUCTION HYPOTHESIS (EXTENDED) $P(m)$ HOLDS AND $P(k)$ HOLDS. THUS BOTH m, k ARE PRODUCTS OF PRIMES, $n+1$ ALSO IS.

GIVES AN ALGORITHM!

HW 3.2: 8, 12, 22, 32, 38, 48