LEC. 11 09/20/99

MATRICES

RECTANGULAR ARRAY

$$\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \cdot & \cdot & \cdot & \cdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}$$

m = THE NUMBER OF ROWS

n = THE NUMBER OF COLUMNS

$m \times n$ MATRIX

$\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}$ IS 2×3; $\begin{bmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{bmatrix}$ IS 3×2

$\begin{bmatrix} 1 \\ 2 \\ 3 \\ -1 \end{bmatrix}$ IS 4×1.

NOTATION: $A = [a_{ij}]$

ROW INDEX

COLUMN INDEX

$A+B=?$ $\quad A=[a_{ij}]$, $B=[b_{ij}]$

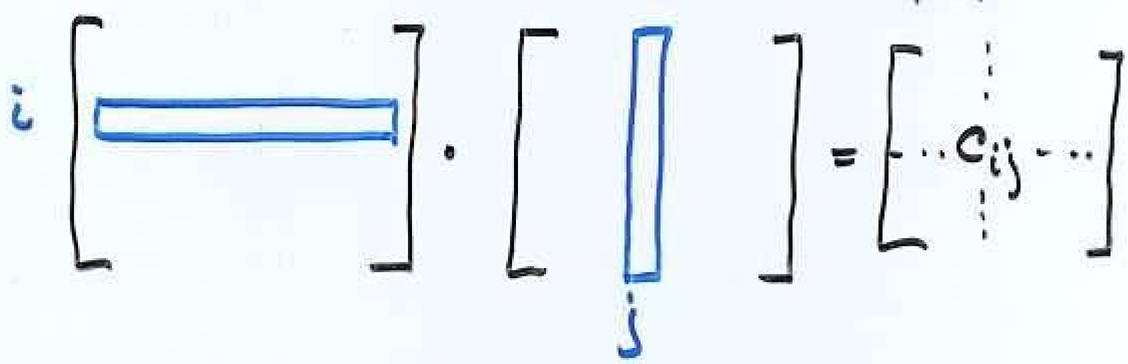
BOTH $m \times n$ $\quad A+B=[a_{ij}+b_{ij}]$

$$\begin{bmatrix} 1 & 2 & 3 \\ 1 & -2 & 3 \\ -1 & 2 & -3 \end{bmatrix} + \begin{bmatrix} 0 & 1 & 3 \\ -1 & 0 & 1 \\ -2 & 2 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 3 & 6 \\ 0 & -2 & 4 \\ -3 & 4 & -3 \end{bmatrix}$$

$A \cdot B = C$ $\quad A$ IS $m \times n$; B IS $n \times p$

ROW-COLUMN RULE: $c_{ij} = \begin{bmatrix} i\text{-th ROW} \\ \text{IN } A \end{bmatrix} \times \begin{bmatrix} j\text{-th} \\ \text{COLUMN} \\ \text{IN } B \end{bmatrix} =$

$$c_{ij} = a_{i1}b_{1j} + a_{i2}b_{2j} + \dots + a_{in} \cdot b_{nj} = \sum_{k=1}^{n} a_{ik} \cdot b_{kj}$$



$$\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \cdot \begin{bmatrix} 2 & 1 \\ 4 & 3 \end{bmatrix} = \begin{bmatrix} 10 & 7 \\ 22 & 15 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 2 \\ -2 & -1 \\ 0 & 3 \end{bmatrix} \cdot \begin{bmatrix} -1 & 2 & 0 \\ 0 & -3 & 1 \end{bmatrix} = \begin{bmatrix} -1 & -4 & 2 \\ 2 & -1 & -1 \\ 0 & -9 & 3 \end{bmatrix}$$

NOT COMMUTATIVE:

$$A \cdot B \neq B \cdot A$$

$$\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \cdot \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \cdot \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$$

$$A \cdot B = 0 \not\Rightarrow A = 0 \vee B = 0$$

NO CANCELLATION RULE:

$$A \cdot B = A \cdot C \wedge A \neq 0 \not\Rightarrow B = C$$

$$\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \cdot \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \cdot \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

PROCEDURE MATRIX MULTIPLICATION(A,B: MATRICES)
FOR $i := 1$ TO m
BEGIN
    FOR $j := 1$ TO n
    BEGIN
        $c_{ij} := 0$
        FOR $q := 1$ TO k
            $c_{ij} := c_{ij} + a_{iq} \cdot b_{qj}$
    END
END $\{C = [c_{ij}]$ IS THE PRODUCT OF A AND B$\}$

COMPLEXITY (NUMBER OF ADDITIONS AND MULTIPLICATIONS): OF $n \times n$ MATRIX MULTIPLICATION

FOR EACH ENTRY $c_{ij} = a_{i1} b_{1j} + \cdots + a_{in} b_{nj}$

n MULTIPLICATIONS, $n-1$ ADDITIONS

TOTAL OPERATIONS $n^2(n-1)$ ADDITIONS

n^3 MULTIPLICATIONS

<u>EXAMPLE</u>. (ASSUME THAT $(A \cdot B) \cdot C = A \cdot (B \cdot C)$)

A IS 30×20, B IS 20×40, C IS 40×10

$A \cdot B$ IS 30×40; $B \cdot C$ IS 20×10

$(A \cdot B)C$ - 36000 MULTIPLICATIONS; $A \cdot (B \cdot C)$ - 14000

$$I_n = \begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ 0 & 0 & \cdots & 1 \end{bmatrix} \quad \text{IDENTITY MATRIX}$$

$$A \cdot I_n = I_n \cdot A = A \qquad (A \text{ IS } n \times n)$$

$$A^0 = I_n, \quad A^k = \underbrace{A \cdot A \cdot \cdots \cdot A}_{k \text{ TIMES}}$$

A^T – TRANSPOSE OF A: $[a_{ij}]^T = [a_{ji}]$

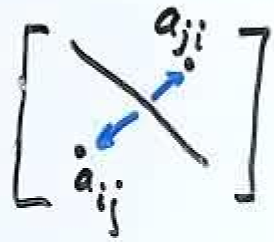
(A IS $m \times n$)

$$\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}^T = \begin{bmatrix} 1 & 4 \\ 2 & 5 \\ 3 & 6 \end{bmatrix}$$

A IS <u>SYMMETRIC</u> IF $A^T = A$

$n \times n$

$$a_{ij} = a_{ji}$$



$$\begin{bmatrix} 1 & 2 & 3 \\ 2 & -1 & 4 \\ 3 & 4 & 0 \end{bmatrix}^T = \begin{bmatrix} 1 & 2 & 3 \\ 2 & -1 & 4 \\ 3 & 4 & 0 \end{bmatrix}$$

ZERO-ONE MATRICES $a_{ij} = \begin{cases} 0 \\ 1 \end{cases}$

$$x \wedge y = \begin{cases} 1, & \text{IF } x = y = 1 \\ 0, & \text{O.W.} \end{cases} ; \quad x \vee y = \begin{cases} 0, & \text{IF } x = y = 0 \\ 1, & \text{O.W.} \end{cases}$$

BOOLEAN PRODUCT

BOOLEAN SUM

$A \vee B \qquad c_{ij} = a_{ij} \vee b_{ij}$

JOIN OF A AND B

$A \wedge B \qquad c_{ij} = a_{ij} \wedge b_{ij}$

MEET OF A AND B

$A \odot B \qquad c_{ij} = (a_{i1} \wedge b_{1j}) \vee \dots \vee (a_{in} \wedge b_{nj})$

BOOLEAN PRODUCT

LIKE $\cdot$ LIKE $+$

$$A^{[0]} = I_n; \quad A^{[r]} = \underbrace{A \odot \dots \odot A}_{r \text{ TIMES}}$$

$$A = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 0 & 0 \\ 1 & 1 & 0 \end{bmatrix}; \quad A^{[2]} = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 0 & 1 \end{bmatrix}; \quad A^{[3]} = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$$

MONOTONICITY! FOR ALL $r \geq 3$ $A = [1]$

HW 2.6: 2a 4b 18 24b 30