CS280 Fall 2001 Prelim 2 Solutions

1. (5 points each) The English alphabet contains 5 vowels and 21 consonants.

a) How many strings of 6 lowercase English letters contain exactly 2 vowels if each letter may be used as often as you like?

There are $C(6,2)$ ways to choose the 2 slots for the vowels, then there are 5 choices for each vowel and 21 choices for each consonant, so there are $C(6,2)5^2 21^4$ total possible strings.

b) How many strings of 6 lowercase English letters contain exactly 2 vowels if no letter is allowed to be used more than once?

Again there are $C(6,2)$ choices for the slots for the vowels, then $P(5,2)$ choices for the vowels and $P(21,4)$ choices for the consonants, so a total of $C(6,2)(5 \cdot 4)(21 \cdot 20 \cdot 19 \cdot 18)$ possible strings.

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2. (5 points each) a) What's the minimum number of people that must be chosen to be sure that at least 2 have the same first initial?

Since there are 26 letters, the Pigeonhole principle implies that if we have 27 people then at least 2 must have the same first initial.

b) What's the minimum number of people that must be chosen to be sure that at least 3 have the same birth month and and were born on the same day of the week (Sat, Sun, Mon, etc)?

Now there are $(12)(7) = 84$ slots, so by the Pigeonhole principle, we want to choose n so that $\lceil n/84 \rceil = 3$. The smallest n for which this is true is $n = 2(84) + 1 = 169$.

c) Suppose there are 50 people with ages between 1 and 98 (1 and 98 are allowed). Show that either there are 2 people with the same age or two whose ages are consecutive integers.

Create 49 slots by grouping the integers into consecutive pairs starting with 1: $\{1, 2\}$,$\{3, 4\}, \ldots \{97, 98\}$. Since there are 50 people with ages contained between 1 and 98, the Pigeonhole principle implies that at least one slot contains at least 2 people. For this slot, either the two people have the same age, or their ages are consecutive integers.

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3. (5 points each) a) How many bit strings contain exactly 6 0's and 9 1's if every 0 must be immediately followed by a 1?

Think of the string '01' as a unit, in which case there are 6 units of '01' and 3 more units of '1'. Hence there are 9 total slots to be filled, and 3 of these are filled with '1', so there are $C(9,3)$ total possible strings.

b) How many solutions are there to the equation $x_1 + x_2 + x_3 + x_4 = 35$, if each x_j is a positive integer (i.e., 1 or bigger)?

This is a stars and bars problem with 4 baskets and 35 stars. However, we need to make sure that each basket has at least one star, so first we set aside 4 stars - one for each basket. Then we have 31 stars and 4 baskets, so there are $C(31 + 4 - 1, 3) = C(34, 3)$ possible ways to assign the stars, so this is the number of solutions.

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4. (5 points each) a) Find the number of elements in $A_1 \cup A_2 \cup A_3$ if there are 100 elements in each set and there are 25 common elements in each pair and 10 elements in the intersection of all 3 sets.

By the inclusion-exclusion principle, we need have $|A_1 \cup A_2 \cup A_3| = |A_1| + |A_2| + |A_3| - (|A_1 \cap A_2| + |A_1 \cap A_3| + |A_2 \cap A_3|) + |A_1 \cap A_2 \cap A_3|$. Using the information given, we get $100 + 100 + 100 - (25 + 25 + 25) - 10 = 235$.

b) Suppose an experiment consists of choosing one of the elements of $A_1 \cup A_2 \cup A_3$ at random with equal probability (where these three sets have the number of elements and intersections as given in part a). Are the events A_1 and A_2 independent?

In order for the events to be independent, we would need $P(A_1 \cap A_2) = P(A_1)P(A_2)$. From part (a), we know there are 235 elements in total, and that A_1 and A_2 each have 100 elements, while $A_1 \cap A_2$ has 25 elements. So the question reduces to comparing $25/235 = .10638$ and $(100/235)^2 = .181077$. Thus the two values are not the same and so the two events are not independent.

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5. (5 points each) The dice in this question are standard six-sided fair dice, and rolling a number with more than one die refers to the sum of the numbers showing on the dice.

a) What is the probability of rolling a 5 with 2 dice?

There are 4 ways to roll a 5: (1,4), (2, 3), (3, 2) and (4, 1). Since there are 36 total equally likely possibilities, the probability of rolling a 5 is $4/36 = 1/9$.

b) What is the probability of rolling a 5 with 3 dice?

There are 6 ways to roll a 5: (1, 1, 3), (1, 2, 2), (1, 3, 1), (2, 1, 2), (2, 2, 1) and (3, 1, 1). Since there are 6^3 total equally likely possibilities, the probability of rolling a 5 is $6/6^3 = 1/36$.

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6. (7 points) What is the expected sum that appears on 2 dice, where each of the dice is biased so that a 3 appears with probability .3 and the other 5 numbers all have equal

probability? (For this problem, do the arithmetic to determine the final probability as a number).

Let X_j be the random variable that records the value of die j for $j = 1, 2$. Then since each of the other numbers appears with probability $(1 - .3)/5 = .14$, we have

$$E(X_j) = .14(1) + .14(2) + .3(3) + .14(4) + .14(5) + .14(6) = .14(18) + .9 = 3.42$$

Since X_1 and X_2 are independent (since the 2 die are independent), we have $E(X_1 + X_2) = E(X_1) + E(X_2) = 6.84$.

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7. a) (8 points) Find a recurrence relation for the number of binary strings of length n that contain 2 consecutive 0's. Also, give the initial conditions for this recurrence relation.

Let a_n denote the number of strings of length n with 2 consecutive 0's. Given a string of length n with 2 consecutive 0's, either it ends in 1 and the first $n-1$ bits have 2 consecutive 0's (and there are a_{n-1} strings of this type), or it ends in 0. If it ends in 0, then the $n-1$st bit may be a 1, in which case the first $n-2$ bits must have 2 consecutive 0's (and there are a_{n-2} strings of this type), or the $n-1$st bit is a 0, in which case the first $n-2$ bits could be anything (and there are 2^{n-2} strings of this type). Hence $a_n = a_{n-1} + a_{n-2} + 2^{n-2}$ for $n \geq 3$. For a string of length 1, there are no strings with 2 consecutive 0's, and for a string of length 2, there is exactly 1, so $a_1 = 0$ and $a_2 = 1$.

b) (5 points) How many strings of length $n = 6$ have 2 consecutive 0's?

From part (a), $a_1 = 0$ and $a_2 = 1$. Then using the recurrence relation, we have

$$a_3 = 1 + 0 + 2 = 3$$

$$a_4 = 3 + 1 + 2^2 = 8$$

$$a_5 = 8 + 3 + 2^3 = 19$$

$$a_6 = 19 + 8 + 2^4 = 43$$

Hence there are 43 such strings.