CS280 Fall 2001 Prelim 1 Solutions

1. (4 points each) Determine which of the following propositions are tautologies:

a) $(p \to q) \leftrightarrow (\neg q \to \neg p)$

Solution:

p	q	$(p \to q)$	$\leftrightarrow$	$(\neg q \to \neg p)$
T	T	T	T	T
T	F	F	T	F
F	T	T	T	T
F	F	T	T	T

Since the value of the expression is true for any value of the variables, this is a tautology.

b) $((p \to q) \to r) \leftrightarrow (p \to (q \to r))$ (Pay attention to the parentheses!)

Solution:

As one example that shows this is not a tautology, set $p =$ F, $q =$ T, and $r =$ F. Then $p \to q$ is T, and r is F, so $((p \to q) \to r)$ is F. But since p is F, $p \to (q \to r)$ is T. Hence the two sides of $\leftrightarrow$ have different values, so the proposition is F, so this is not a tautology.

c) $(s \to p) \vee (\neg p \wedge s)$

Solution:

s	p	$(s \to p)$	$\vee$	$(\neg p \wedge s)$
T	T	T	T	F
T	F	F	T	T
F	T	T	T	F
F	F	T	T	F

Since the value of the proposition is true for any value of the variables, this is a tautology.

2. (4 points each) Determine whether each of the following statements is true or false. The domain of discourse in each case is the real numbers.

a) $\exists x \exists y((x > y) \rightarrow (x^2 < y^2))$

Solution:

This is true. For example, let $x = 0$, $y = -1$; then $x > y$ and $x^2 < y^2$, so the implication is true. As another example, let $x = -1$, $y = 0$; then $x < y$, and the implication is true because the hypothesis is not true.

b) $\forall x \exists z \forall y((x < y) \rightarrow ((z > x) \wedge (z < y)))$

Solution:

This is false. One way to see this is to let $x = 0$ and let z be any number. If $z > x$, then let y be in the interval (x, z). Then $x < y$ but $z > y$, so the implication is false. On the other hand, if $z \leq x$, then let $y > x$. Then again $x < y$, but $z \leq x$, so the implication is false here also. So for this choice of x there does not exist a z so that the proposition is true for all values of y, hence the statement is false.

c) $\forall x(((x > 0) \rightarrow (x^3 < 0)) \vee ((x^3 < 0) \rightarrow (x > 0)))$

Solution:

This is true. If $x > 0$, then $x^3 > 0$, so the first implication is false, but since $x^3 > 0$, the second implication is true since the hypothesis is false, hence the statement is true. On the other hand, if $x \leq 0$, then the first implication is true, so the statement is true. Hence for all x, the statement is true.

3. (5 points each)

a) Let $A = \{\emptyset, \{\emptyset\}\}$. Find $P(A)$, the power set of A.

Solution:

The power set contains all possible subsets of A. In this case, there are only two elements in A, so $P(A)$ has four elements corresponding to the four possible ways of including or excluding two elements.

$P(A) = \{\emptyset, \{\emptyset\}, \{\{\emptyset\}\}, \{\emptyset, \{\emptyset\}\}\}$

b) Show that $(A - C) \cap (C - B) = \emptyset$. You must give a logical argument, not just show Venn diagrams.

Solution:

If $x \in A - C$, then $x \in A$ and $x \notin C$. Likewise, if $x \in C - B$, then $x \in C$ and $x \notin B$. Hence if $x \in (A - C) \cap (C - B)$, then $x \in C$ and $x \notin C$, which is impossible. Hence there is no such x, so $(A - C) \cap (C - B) = \emptyset$.

4. (10 points) Let X and Y be sets and suppose there is an injection (1-to-1 function) $f : X \to Y$ (but f is not necessarily onto). Prove that there is a surjection (onto function) $g : Y \to X$ (i.e, for each $x \in X$, there exists $y \in Y$ so that $g(y) = x$).

Solution:

Let $S = f(X)$. Since f is 1-to-1, we know that for each $s \in S$, there is a unique element $x \in X$ so that $f(x) = s$, or rewriting this, that $f^{-1}(s) = x$. Define $g(s) = f^{-1}(s)$ for $s \in S$. Then g maps S onto X, but g may not be defined on all of Y. So, choose $a \in X$, and define $g(y) = a$ for $y \in Y - S$. Then g is well-defined on all of Y and maps Y onto X.

5. (5 points each)

a) Find $\sum_{n=1}^{200} n$

Solution:

If you don't remember the formula, then write

$$x = 1 + 2 + \cdots + 200$$

$$x = 200 + 199 + \cdots + 1$$

and add to get

$$2x = 201 + 201 + \cdots 201 = 200(201).$$

Hence $x = 200(201)/2 = 20100$.

b) Find $\sum_{n=0}^{25} 3(-3)^n$

Solution:

First factor out the 3. Again, if you don't remember the formula, write $x = 1 + (-3) + \cdots + (-3)^{24} + (-3)^{25}$, then multiply by $1 - (-3)$ to get $x(1 - (-3)) = 1 + (-3) + \cdots + (-3)^{25} - (-3) \cdots - (-3)^{25} - (-3)^{26} = 1 - (-3)^{26}$. Hence

$$x = \frac{1 - (-3)^{26}}{1 - (-3)} = -\frac{3^{26} - 1}{4},$$

so the answer to the problem is $3x = -3\frac{3^{26} - 1}{4}$.

6. a) (2 points each) Which of the following have solutions? Explain why or why not. If there is a solution, use the Euclidean algorithm to find it.

i) $8x \equiv 1 \bmod 12$

Solution:

There is no solution because $\gcd(8, 12) = 4 > 1$.

ii) $7x \equiv 1 \bmod 30$

Solution:

There is a solution because $\gcd(7, 30) = 1$. Using the Euclidean algorithm, we have $30 = 4 \cdot 7 + 2$ and $7 = 3 \cdot 2 + 1$, hence $1 = 7 - 3 \cdot 2 = 7 - 3(30 - 4 \cdot 7) = 13 \cdot 7 - 3 \cdot 30$. Thus $7 \cdot 13 \equiv 1 \bmod 30$, so $x = 13$ is a solution.

iii) $100x \equiv 1 \bmod 102$

Solution:

There is no solutions because $\gcd(100, 102) = 2 > 1$.

b) (5 points) If the product of two integers is $2^7 3^8 5^2 7^{11}$ and their greatest common divisor is $2^3 3^4 5$, what is their least common multiple? Explain.

Solution:

From a theorem from class, given two positive integers a and b, we know that $ab = \gcd(a,b) \cdot \operatorname{lcm}(a,b)$, or $\operatorname{lcm}(a,b) = ab/\gcd(a,b)$. Hence in this case we have a least common multiple of $2^7 3^8 5^2 7^{11}/2^3 3^4 5 = 2^4 3^4 5 7^{11}$.

7. (10 points) Prove by induction that 5 divides $11^n - 6$ for all positive integers n.

Solution:

The base case is $n = 1$, in which case $11^1 - 6 = 5$ is clearly divisible by 5.

For the inductive step, assume that 5 divides $11^n - 6$ for a positive integer n. We want to prove that 5 divides $11^{n+1} - 6$. Expanding, we have

$$\begin{aligned} 11^{n+1} - 6 &= 11 \cdot 11^n - 6 \\ &= 11(11^n - 6) + 66 - 6 \\ &= 11(11^n - 6) + 60. \end{aligned}$$

Since $11^n - 6$ is divisible by 5 by the induction hypothesis, and since 60 is divisible by 5, we see that $11^{n+1} - 6$ is also divisible by 5. By induction, $11^n - 6$ is divisible by 5 for all positive integers n.