CS 280 Prelim 1 Solutions October 7, 1999

Show all your work.

1. (15 points) Determine which of the following propositions are tautologies

a) $(p \to \neg p) \leftrightarrow \neg p$ b) $(p \to \neg q) \leftrightarrow \neg(p \wedge q)$ c) $((\neg p \wedge q) \to r) \to ((\neg q \to p) \to r)$.

Answer:

a) For all a, b, $a \to b$ is always equivalent to $b \vee \neg a$. So $p \to \neg p$ is always equivalent to $\neg p \vee \neg p$, which is clearly equivalent to $\neg p$. (Could also use a truth table).

b) The following truth table shows that $(p \to \neg q) \leftrightarrow \neg(p \wedge q)$ is indeed a tautology.

p	q	$\neg q$	$p \to \neg q$	$\neg(p \wedge q)$	$(p \to \neg q) \leftrightarrow \neg(p \wedge q)$
0	0	1	1	1	1
0	1	0	1	1	1
1	0	1	1	1	1
1	1	0	0	0	1

c) The formula is not a tautology. It suffices to show one row in a truth table that assigns 0 to the whole formula:

p	q	r	$\neg p \wedge q$	$\neg q \to p$	$(\neg p \wedge q) \to r$	$(\neg q \to p) \to r$	$((\neg p \wedge q) \to r) \to ((\neg q \to p) \to r)$
1	0	0	0	1	1	0	0

2. (15 points)

a) Establish the logical equivalence of $\neg \forall x(A \to B)$ and $\exists x(A \wedge \neg B)$.

Answer: Put $S \equiv \neg \forall x(A \to B)$. From the truth table, we know that $A \to B$ is equivalent to $\neg A \vee B$, so S is equivalent to $\neg \forall x(\neg A \vee B)$. Also, by Table 3, p.33, we know this is equivalent to $\exists x \neg(\neg A \vee B)$. Now using de Morgan's rules, this becomes $\exists x(A \wedge \neg B)$.

b) Show that $\exists x(A(x) \wedge B(x))$ and $\exists x A(x) \wedge \exists x B(x)$ are not logically equivalent.

Answer: Let the universe of discourse be $\mathcal{Z}$, the set of integers. Let $A(x) \equiv$ "x is positive," and let $B(x) \equiv$ "x is negative." Then the first statement says that there exists an integer which is both positive and negative, which is false. The second statement says that there exists a positive integer and there exists a negative integer, which is true. So the two are not equivalent.

3. (15 points) The *composition* of functions f and g, denoted by $f \circ g$, is defined by $(f \circ g)(a) = f(g(a))$. The *inverse* of h is the function h^{-1} such that $h^{-1} \circ h$ and $h \circ h^{-1}$ are identity functions, i.e. $(h^{-1} \circ h)(a) = a$ and $(h \circ h^{-1})(b) = b$ for all a from the domain of h and all b from the codomain of h.

a) Give an example of f and g such that $f \circ g$ and $g \circ f$ are different

Answer: Put $f(x) = x + 1$, and $g(x) = x^2$. Then $(f \circ g)(x) = x^2 + 1$, whereas $(g \circ f)(x) = (x + 1)^2$.

b) Suppose f and g are invertible. Show that $(f \circ g)^{-1}$ equals to $g^{-1} \circ f^{-1}$.

Answer: It suffices to demonstrate that

$$((f \circ g) \circ (g^{-1} \circ f^{-1}))(x) = x \quad \text{and} \quad ((g^{-1} \circ f^{-1}) \circ (f \circ g))(y) = y.$$

The cases are similar. We consider the first one only.

$$\begin{aligned}
((f \circ g) \circ (g^{-1} \circ f^{-1}))(x) &= (f \circ g)((g^{-1} \circ f^{-1})(x)) \\
&= f(g((g^{-1} \circ f^{-1})(x))) \\
&= f(g(g^{-1}(f^{-1}(x)))) \\
&= f((g \circ g^{-1})(f^{-1}(x))) \\
&= f(f^{-1}(x)) \\
&= (f \circ f^{-1})(x) \\
&= x
\end{aligned}$$

4. (10 points)

a) How many multiplications does the standard row-column algorithm uses to compute the product of an $m \times n$ matrix and an $n \times p$ matrix? Explain why.

Answer: If A is $m \times n$ and B is $n \times p$, and $C = A \cdot B$, then

$$C_{ij} = \sum_{k=1}^{n} A_{ik} B_{kj}.$$

This formula involves n multiplications for each entry C_{ij}. The matrix C has size mp, i.e., there are mp entries C_{ij}, so the total number of multiplications is mnp.

b) Suppose you have to find $A \cdot B \cdot C$, were A is a 3×10 matrix, B - 10×50 matrix and C - 50×2 matrix. Which order of multiplication should you choose: $(A \cdot B) \cdot C$ or $A \cdot (B \cdot C)$?

Answer: Count the number of multiplications in both ways:

$(A \cdot B) \cdot C$ requires $(3 \cdot 10 \cdot 50) + (3 \cdot 50 \cdot 2) = 1800$

$A \cdot (B \cdot C)$ requires $(10 \cdot 50 \cdot 2) + (3 \cdot 10 \cdot 2) = 1060$

The second way is faster.

5. (15 points) Compute the greatest common divisor (gcd) of 156 and 93. Find integers x and y such that $156x + 93y = \gcd(156, 93)$.

Answer. Use Euclid's algorithm:

$$\begin{aligned} 156 &= 1 \cdot 93 + 63 \\ 93 &= 1 \cdot 63 + 30 \\ 63 &= 2 \cdot 30 + 3 \\ 30 &= 10 \cdot 3 \end{aligned}$$

So gcd(156,93) = 3.

From the third division above, we can write $3 = 63 - 2 \cdot 30$. From the second division, we can write $3 = 63 - 2 \cdot (93 - 63) = 3 \cdot 63 - 2 \cdot 93$. And from the first division, we can write $3 = 3 \cdot (156 - 93) - 2 \cdot 93 = 3 \cdot 156 - 5 \cdot 93$. So $x = 3$ and $y = -5$.

6. (10 points)
a) Find the base 8 expansion of $(123)_{10}$.
Answer. $123 = 1 \cdot 64 + 7 \cdot 8 + 3 \cdot 1$, so $(123)_8 = 173$.

b) Find the binary expansion of $(123)_{10}$
Answer. Look at part (a). Each digit in the octal representation can be represented using three binary digits. So $(1)_8 = (001)_2$, $(7)_8 = (111)_2$, and $(3)_8 = (011)_2$. Now concatenate them: $(123)_10 = (173)_8 = (001111011)_2$. Eliminating the leading 0's, the answer is $(1111011)_2$.

7. (20 points) By the Chinese Remainder Theorem for each integers a, b and c ($0 \le a < 9$, $0 \le b < 10$ and $0 \le c < 11$) there is a unique nonnegative integer $x < 990 = 9 \cdot 10 \cdot 11$ such that $x \equiv a(\text{mod} 9)$, $x \equiv b(\text{mod} 10)$ and $x \equiv c(\text{mod} 11)$.

a) Find such a, b and c for $x = 801$.
Answer. Dividing by 9, 10, and 11, we find $a = 0$, $b = 1$, $c = 9$.

b) Find an positive integer x satisfying $x \equiv 1(\text{mod} 9)$, $x \equiv 0(\text{mod} 10)$ and $x \equiv 1(\text{mod} 11)$
Answer. We use the method described on p. 142, which gives a formula

$$x \equiv aM_1y_1 + bM_2y_2 + cM_3y_3 \pmod{m}$$

Here $a = 1$, $b = 0$, $c = 1$, $m = 9 \cdot 10 \cdot 11$, $M_1 = 10 \cdot 11 = 110$, $M_2 = 9 \cdot 11 = 99$, $M_3 = 9 \cdot 10 = 90$. The numbers y_1, y_2 and y_3 (so-called inverses) can be defined by:

$$\begin{aligned} 110y_1 &\equiv 1 \pmod{9} \\ 99y_2 &\equiv 1 \pmod{10} \\ 90y_3 &\equiv 1 \pmod{11} \end{aligned}$$

Use the Euclidean algorithms to find the inverses:

- For y_1, we have

$$\begin{aligned} 110 &= 12 \cdot 9 + 2 \\ 9 &= 4 \cdot 2 + 1 \end{aligned}$$

  This means $1 = 9 - 4 \cdot 2 = 9 - 4 \cdot (110 - 12 \cdot 9) = 49 \cdot 9 - 4 \cdot 110$. So $y_1 = -4 \equiv 5 \pmod 9$.

- For y_3, we have

$$\begin{aligned} 90 &= 8 \cdot 11 + 2 \\ 11 &= 5 \cdot 2 + 1 \end{aligned}$$

  This means $1 = 11 - 5 \cdot 2 = 11 - 5 \cdot (90 - 8 \cdot 11) = 41 \cdot 11 - 5 \cdot 90$. So $y_3 = -5 \equiv 6 \pmod{11}$.

Finally

$$\begin{aligned} x &\equiv aM_1y_1 + bM_2y_2 + cM_3y_3 \pmod{990} \\ &\equiv 1 \cdot 110 \cdot 5 + 0 + 1 \cdot 90 \cdot 6 \pmod{990} \\ &\equiv 550 + 540 \pmod{990} \\ &\equiv 1090 \pmod{990} \\ &\equiv 100 \pmod{990} \end{aligned}$$

We may take $x = 100$.