CS 280 – Discrete Structures
Solutions to Assignment 2

Section 1.4

$\boxed{10}$

For example: $A = \{a\}$, $B = \{\{a\}, a\}$

$\boxed{14}$

Yes. For finite sets, we can reason as follows: Consider two sets A and B that have identical power sets $P(A)$ and $P(B)$. From the definition of *power set* it follows that both $P(A)$ and $P(B)$ contain exactly one largest set, namely A and B. Since the power sets are equal, A and B must be equal as well. To see this, simply note that if they are not (but $P(A)$ and $P(B)$ are) it means that $A = C$ where C is some other set in $P(B)$. So the cardinality of A is the same as the cardinality of C. There is no subset in $P(A)$ that has a larger cardinality than A, while the opposite is true for C in $P(B)$. Therefore $P(B)$ and $P(A)$ can not have the same elements, violating the original assumption.

For infinite sets, it's easier to use the contrapositive, that is, to prove that if A and B are not equal , then P(A) and P(B) are not equal. So suppose that A and B are not equal. Then either there is an element in A that is not in B or there is an element in B that is not in A. Starting with the first case, let *a* be an element in A but not B. Then the set $\{a\}$ is in the power set of A but cannot be in the power set of B. Hence the two power sets are different. Likewise, if there is an element of B that is not in A, then again the power sets are different. Hence, if the power sets of A and B are the same, then the sets A and B must be identical.

Section 1.5

$\boxed{8}$

$A = \{3,6,9,1,5,7,8\}, B = \{3,6,9,2,10\}$. Start with their intersection and then add "whatever is missing" of which we learn in the differences of the two sets.

$\boxed{10}$

(a) If $x \in (A \cap B)$ then $x \in A \wedge x \in B$, hence $x \in A$.

(b) If $x \in A$ then $(x \in A) \vee (x \in B)$, hence $x \in (A \cup B)$.

(c) If $x \in (A - B)$ then $x \in A \wedge x \notin B$, hence $x \in A$.

(d) The left side of the equation is exactly $\{x \mid x \in A \wedge x \in (B - A)\}$ which translates to $\{x \mid x \in A \wedge (x \in B \wedge x \notin A))\}$ or $\{x \mid x \in A \wedge x \notin A \wedge x \in B\}$. No such x exists, therefore this expression is equivalent to the empty set.

(e) The left side of the equation is exactly $\{x \mid x \in A \vee x \in (B - A)\}$ which translates to $\{x \mid x \in A \vee (x \in B \wedge x \notin A))\}$. The right side is $\{x \mid x \in A \vee x \in B\}$. Suppose $a \in LHS$. Then, $a \in A$ places it into the *RHS.* The alternative is that $a \notin A$. But then we know that since $a \in LHS$, $a \in B$, and thus is also in the *RHS.* So $LHS \subseteq RHS$. Now suppose $a \in RHS$. Again, if $a \in A$, it is immediately in the *LHS.* The alternative is that $a \notin A$, but than since $a \in RHS$, $a \in B$ so, $x \in B \wedge x \notin A$. Therefore $LHS \supseteq RHS$. We thus conclude $LHS = RHS$.

Section 1.6

$\boxed{4}$

a) Domain: all nonnegative integers. Range: integers 0-9.
b) Domain: all positive integers. Range: all positive integers except for the first one.
c) Domain: all bit strings. Range: naturals.
d) Domain: all bit strings. Range: naturals - {0} (we assume strings can not be empty).

$\boxed{20}$

Yes. Let $f : M \to N$ be one-to-one. Let $g : L \to M$ be another function and we know $f \circ g$ is one-to-one. Suppose that *g* is not one-to-one. Then, from the definition of injection (p.59) it follows that for some *x* and *y* in *L*, $g(x) = g(y)$ and $x \neq y$. So, we have $f(g(x)) = f(g(y))$ since *f* is a function. This can be written as $(f \circ g)(x) = (f \circ g)(y)$. From the fact that the compound function is injective it follows that $x = y$, thus leading o a contradiction. The original assumption must therefore be flawed.

$\boxed{26}$

(a)

Suppose $x \in f(S \cup T)$

$x \in \{f(a) \mid a \in (S \cup T)\}$

$x \in \{f(a) \mid a \in \{i \mid i \in S \vee i \in T\}\}$

$x \in \{f(a) \mid a \in S \vee a \in T\}$.

$x \in \{f(a) \mid a \in S\} \vee x \in \{f(a) \mid a \in T\}$

$x \in f(S) \vee x \in f(T)$ or

$x \in f(S) \cup f(T)$.

Conversely, $x \in f(S) \cup f(T)$.

$x \in f(S) \vee x \in f(T)$

$x \in \{f(a) \mid a \in S\} \vee x \in \{f(a) \mid a \in T\}$

$x \in \{f(a) \mid a \in S \vee a \in T\}$

$x \in \{f(a) \mid a \in (S \cup T)\}$

$x \in f(S \cup T)$.

So, $f(S \cup T) \subseteq f(S) \cup f(T)$ and $f(S \cup T) \supseteq f(S) \cup f(T)$

$f(S \cup T) = f(S) \cup f(T)$

(b)

Suppose $x \in f(S \cap T)$.

$x \in \{f(a) \mid a \in (S \cap T)\}$

$x \in \{f(a) \mid a \in \{i \mid i \in S \wedge i \in T\}\}$

$x \in \{f(a) \mid a \in S \wedge a \in T\}$.

But $\{f(a) \mid a \in S \wedge a \in T\} \subseteq \{f(a) \mid a \in S\} = f(S)$ and

$\{f(a) \mid a \in S \wedge a \in T\} \subseteq \{f(a) \mid a \in T\} = f(T)$

So, $x \in f(S) \wedge x \in f(T)$.

$x \in f(S) \cap f(T)$

$f(S \cap T) \subseteq f(S) \cap f(T)$

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There is an answer in the back of the book, where $S \cap T = \varnothing$.
But consider the case where this is not so:
Let $S = \{a,b\}, T = \{b,c\}, f(a) = f(c) = x, f(b) = y$. Furthermore, assume that a,b,c,x,y are all distinct. Then we have

$$f(S) = \{x,y\}, f(T) = \{x,y\}, S \cap T = \{b\}, f(S \cap T) = y, f(S) \cap f(T) = \{x,y\}$$

So, $f(S \cap T) \subset f(S) \cap f(T)$.

Section 1.7

$\boxed{6}$

(f) 1, 3, 7, 15, 31, 63, 127, 255, 511, 1023 (n=1 to 10)
(g) 1, 2, 2, 4, 8, 11, 33, 37, 148, 153
(h) 1, 2, 2, 2, 2, 3, 3, 3, 3, 3

$\boxed{10}$

(b) Start with 7. Obtain each new element by adding 4 to the previous one.

(d) There are many possibilities for this one. Here is one: Let a_1 be 1, a_2 be 2, and $a_{n+2} = a_{n+1} + a_n$ for n>2. To form the given sequence, use 1 copy of a_1, 3 copies of a_2, 5 copies of a_3, and in general, $2k-1$ copies of a_k to the sequence.

(f) Start with $n = 1$. The nth $(n > 1)$ term a_n is $a_{n-1}(2n-1)$. In other words, multiply the previous element by the next odd number

(h) The nth member of the sequence is $2^{2^{n-1}}$

$\boxed{20}$

$$\sum_{k=1}^{n} \tfrac{1}{k(k+1)} = \sum_{k=1}^{n} (\tfrac{1}{k} - \tfrac{1}{(k+1)}) = \sum_{k=1}^{n} (-1)(\tfrac{1}{k+1} - \tfrac{1}{k}) = -\sum_{k=1}^{n} (\tfrac{1}{k+1} - \tfrac{1}{k})$$

If we let a_k denote $\frac{1}{k+1}$, then the above is equivalent to $-\sum_{k=1}^{n} (a_k - a_{k-1})$

$$= -(a_n - a_0) = -(\tfrac{1}{n+1} - 1) = \tfrac{n+1}{n+1} - \tfrac{1}{n+1} = \tfrac{n}{n+1}.$$

$\boxed{32}$

(a) **Countable**. Pick the first two naturals (starting with 1) and add them and their negatives to a list of suitable numbers. Skip one natural. Repeat the procedure. Thus we have 1,-1,2,-2,4,-4,5,-5,7,-7,8,-8,... In this way we establish a sequence.

b) **Countable**. Start with 0. Keep adding 5 to obtain a new element and alternate the sign. Ignore every 7'th pair thus produced. Thus we have a sequence:
5,-5,10,-10,15,-15, 20,-20,25,-25,30,-30,40,-40,...

c) **Countable**. Some reals have infinite number of decimal 1's after the decimal point. However, since we are restricted to only one choice of "characters" in that infinity, all of such numbers differ only by what happens before the decimal point, and that string is always finite, of course. Therefore, an ordering can be found. For example, starting with $n = 0$, pick all strings of 1's with length n. We then have $n+1$

ways of selecting a decimal point in that string, and 2 ways of selecting a sign (+ or -). We then also have 2 ways of selecting an infinite string with a finite (*n*-sized) prefix before the decimal point (2 because of the sign). We thus establish a sequence:

let INF denote 1111.... an infinite string of 1's

.INF, -.INF, 1, -1, .1, -.1, 1.INF, -1.INF, 11, 1.1, .11, -11, -1.1, -.11, 11.INF, -11.INF, ...

d) **Not Countable**. Use Diagonolization. Suppose the set is countable. Consider a subset of that countable set. Namely the numbers that have an infinite string representation of 1 and 9, with decimal in front (i.e. numbers between .11111111..... and .9999999999.....) Pick a correspondence between these and the naturals. Then note that number that lies on the diagonal with 1's and 9's interchanged is never encountered in the list.

Consider the "table" below:

1 **1**91911991919111199...
2 9**1**9999199191991999 1...
3 11**9**911199999199919...
4…

The diagonal number is 119... . The number 991... is never in the list (it would violate the diagonal). This leads to a contradiction.