0!

It's useful to define $0! = 1$.

Why?

1. Then we can inductively define
$$(n+1)! = (n+1)n!,$$
and this definition works even taking 0 as the base case instead of 1.
2. A better reason: Things work out right for $P(n,0)$ and $C(n,0)$!

How many permutations of n things from n are there?

$$P(n,n) = \frac{n!}{(n-n)!} = \frac{n!}{0!} = n!$$

How many ways are there of choosing n out of n? 0 out of n?

$$\binom{n}{n} = \frac{n!}{n!0!} = 1$$
$$\binom{n}{0} = \frac{n!}{0!n!} = 1$$

More Questions

Q: How many ways are there of choosing k things from $\{1,\ldots,n\}$ if 1 and 2 can't both be chosen? (Suppose $n, k \geq 2$.)

A: First find all the ways of choosing k things from n—$C(n,k)$. Then subtract the number of those ways in which both 1 and 2 are chosen:

- This amounts to choosing $k-2$ things from $\{3,\ldots,n\}$: $C(n-2,k-2)$.

Thus, the answer is

$$C(n,k) - C(n-2,k-2)$$

Q: What if order matters?

A: Have to compute how many ways there are of picking k things, two of which are 1 and 2.

$$P(n,k) - k(k-1)P(n-2,k-2)$$

Q: How many ways are there to distribute four distinct balls evenly between two distinct boxes (two balls go in each box)?

A: All you need to decide is which balls go in the first box.
$$C(4,2) = 6$$

Q: What if the boxes are indistinguishable?

A: $C(4,2)/2 = 3$.

Combinatorial Identities

There all lots of identities that you can form using $C(n,k)$. They seem mysterious at first, but there's usually a good reason for them.

Theorem 1: If $0 \leq k \leq n$, then
$$C(n,k) = C(n,n-k).$$

Proof:
$$C(n,k) = \frac{n!}{k!(n-k)!} = \frac{n!}{(n-k)!(n-(n-k))!} = C(n,n-k)$$

Q: Why should choosing k things out of n be the same as choosing $n-k$ things out of n?

A: There's a 1-1 correspondence. For every way of choosing k things out of n, look at the things not chosen: that's a way of choosing $n-k$ things out of n.

This is a better way of thinking about Theorem 1 than the combinatorial proof.

Theorem 2: If $0 < k < n$ then

$$\binom{n}{k} = \binom{n-1}{k} + \binom{n-1}{k-1}$$

Proof 1: (Combinatorial) Suppose we want to choose k objects out of $\{1, \ldots, n\}$. Either we choose the last one (n) or we don't.

1. How many ways are there of choosing k without choosing the last one? $C(n-1, k)$.
2. How many ways are there of choosing k including n? This means choosing $k-1$ out of $\{1, \ldots, n-1\}$: $C(n-1, k-1)$.

Proof 2: Algebraic ...

Note: If we define $C(n, k) = 0$ for $k > n$ and $k < 0$, Theorems 1 and 2 still hold.

Pascal's Triangle

Starting with $n = 0$, the nth row has $n+1$ elements:

$$C(n, 0), \ldots, C(n, n)$$

Note how Pascal's Triangle illustrates Theorems 1 and 2.

Theorem 3: For all $n \geq 0$:

$$\sum_{k=0}^{n} \binom{n}{k} = 2^n$$

Proof 1: $\binom{n}{k}$ tells you all the way of choosing a subset of size k from a set of size n. This means that the LHS is *all* the ways of choosing a subset from a set of size n. The product rule says that this is 2^n.

Proof 2: By induction. Let $P(n)$ be the statement of the theorem.

Basis: $\Sigma_{k=0}^{0} \binom{0}{k} = \binom{0}{0} = 1 = 2^0$. Thus $P(0)$ is true.

Inductive step: How do we express $\Sigma_{k=0}^{n} C(n, k)$ in terms of $n-1$, so that we can apply the inductive hypothesis?

- Use Theorem 2!

Theorem 4: For any nonnegative integer n

$$\sum_{k=0}^{n} k\binom{n}{k} = n2^{n-1}$$

Proof 1:

$$\begin{aligned}
& \Sigma_{k=0}^{n} k\binom{n}{k} \\
= & \Sigma_{k=1}^{n} k\frac{n!}{(n-k)!k!} \\
= & \Sigma_{k=1}^{n} \frac{n!}{(n-k)!(k-1)!} \\
= & n\,\Sigma_{k=1}^{n} \frac{(n-1)!}{(n-k)!(k-1)!} \\
= & n\,\Sigma_{k=1}^{n} \frac{(n-1)!}{(n-k)!(k-1)!} \\
= & n\,\Sigma_{k=0}^{n-1} \frac{(n-1)!}{(n-1-k)!k!} \\
= & n\,\Sigma_{k=0}^{n-1} C(n-1, k) \\
= & n2^{n-1}
\end{aligned}$$

Proof 2: LHS tells you all the ways of picking a subset of k elements out of n (a subcommittee) and designating one of its members as special (subcommittee chairman).

What's another way of doing this? Pick the chairman first, and then the rest of the subcommittee!

Theorem 5:

$$(n-k)\binom{n}{k} = (k+1)\binom{n}{(k+1)} = n\binom{(n-1)}{k}$$

Theorem 6:

$$C(n,k)C(n-k,j) = C(n,j)C(n-j,k) = C(n,k+j)C(k+j,j)$$

Theorem 7: $P(n,k) = nP(n-1,k-1)$.

The Binomial Theorem

We want to compute $(x+y)^n$.
Some examples:

$$(x+y)^1 = x+y$$
$$(x+y)^2 = x^2+2xy+y^2$$
$$(x+y)^3 = x^3+3x^2y+3xy^2+y^3$$
$$(x+y)^4 = x^4+4x^3y+6x^2y^2+4xy^3+y^4$$

The pattern of the coefficients is just like that in the corresponding row of Pascal's triangle!

Binomial Theorem:

$$(x+y)^n = \sum_{k=0}^{n}\binom{n}{k}x^{n-k}y^k$$

Proof 1: By induction on n. $P(n)$ is the statement of the theorem.

Basis: $P(1)$ is obviously OK. (So is $P(0)$.)

Inductive step:

$$\begin{aligned}
& (x+y)^{n+1} \\
= {} & (x+y)(x+y)^n \\
= {} & (x+y)\,\Sigma_{k=0}^{n}\tbinom{n}{k}x^{n-k}y^k \\
= {} & \Sigma_{k=0}^{n}\tbinom{n}{k}x^{n-k+1}y^k + \Sigma_{k=0}^{n}\tbinom{n}{k}x^{n-k}y^{k+1} \\
= {} & \ldots \qquad \text{[Lots of missing steps]} \\
= {} & y^{n+1} + \Sigma_{k=0}^{n}(\tbinom{n}{k}+\tbinom{n}{k-1})x^{n-k+1}y^k \\
= {} & y^{n+1} + \Sigma_{k=0}^{n}(\tbinom{n+1}{k}x^{n+1-k}y^k \\
= {} & \Sigma_{k=0}^{n+1}\tbinom{n+1}{k}x^{n+1-k}y^k
\end{aligned}$$

Proof 2: What is the coefficient of the $x^{n-k}y^k$ term in $(x+y)^n$?

Using the Binomial Theorem

Q: What is $(x+2)^4$?

A:

$$\begin{aligned}
& (x+2)^4 \\
= {} & x^4 + C(4,1)x^3(2) + C(4,2)x^2 2^2 + C(4,3)x2^3 + 2^4 \\
= {} & x^4 + 8x^3 + 24x^2 + 32x + 16
\end{aligned}$$

Q: What is $(1.02)^7$ to 4 decimal places?

A:

$$\begin{aligned}
& (1+.02)^7 \\
= {} & 1^7 + C(7,1)1^6(.02) + C(7,2)1^5(.0004) + C(7,3)(.000008) + \cdot \\
= {} & 1 + .14 + .0084 + .00028 + \cdots \\
\approx {} & 1.14868 \\
\approx {} & 1.1487
\end{aligned}$$

Note that we have to go to 5 decimal places to compute the answer to 4 decimal places.

In the book they talk about the *multinomial theorem*. That's for dealing with $(x+y+z)^n$.

They also talk about the *binomial series theorem*. That's for dealing with $(x+y)^\alpha$, when α is any *real* number (like 0.3).

You're not responsible for these results.

Balls and Urns

"Balls and urns" problems are paradigmatic. Many problems can be recast as balls and urns problems, once we figure out which are the balls and which are the urns.

How many ways are there of putting b balls into u urns?

- That depends whether the balls are distinguishable and whether the urns are distinguishable

How many ways are there of putting 5 balls into 2 urns?

- If both balls and urns are distinguishable: $2^5 = 32$
  - Choose the subset of balls that goes into the first urn
  - Alternatively, for each ball, decide which urn it goes in
  - This assumes that it's OK to have 0 balls in an urn.

- If urns are distinguishable but balls aren't: 6
  - Decide how many balls go into the first urn: 0, 1, ..., 5
- If balls are distinguishable but urns aren't: $2^5/2 = 16$
- If balls and urns are indistinguishable: $6/2 = 3$

What if we had 6 balls and 2 urns?

- If balls and urns are distinguishable: 2^6
- If urns are distinguishable and balls aren't: 7
- If balls are distinguishable but urns aren't:

$$2^6/2 = 2^5$$

- If balls and urns are indistinguishable: 4
  - It can't be $7/2$, since that's not an integer
  - The problem is that if there are 3 balls in each urn, and you switch urns, then you get the same solution

Reducing Problems to Balls and Urns

Q1: How many different configurations are there in Towers of Hanoi with n rings?

A: The urns are the poles, the balls are the rings. Both are distinguishable.

Q2: How many solutions are there to the equation $x+y+z=65$, if x, y, z are nonnegative integers?

A: You have 65 indistinguishable balls, and want to put them into 3 distinguishable urns (x, y, z). Each way of doing so corresponds to one solution.

Q3: How many ways can 8 electrons be assigned to 4 energy states?

A: The electrons are the balls; they're indistinguishable. The energy states are the urns; they're distinguishable.