CS 222: Introduction to Scientific Computing
Spring 2001
Practice Final Exam

Handed out: Wed, May 2 on the web.

This examination lasted 135 minutes and had 135 points total. It was closed book and closed note, but students were permitted to use an $8\frac{1}{2}'' \times 11''$ crib-sheet with notes written on both sides.

1. [**5 points**] What is the meaning of a "stable" algorithm? Full credit will be given for any one of the definitions of "stable" that was covered in lecture.

2. [**5 points**] Describe an algorithm to determine whether a symmetric square matrix is positive definite, and indicate how many flops this algorithm requires.

3. [**5 points**] How many flops (accurate to the leading term) are required to compute A^TA, where A is an $m \times 2$ matrix, accurate to the leading term (in m)?

4. [**5 points**] In Newton's method for multivariate optimization, what should be done if the Hessian is not positive definite? (Name one reasonable course of action.)

5. [**5 points**] Consider a one-variable function $f : \mathbf{R} \rightarrow \mathbf{R}$. Let x^* be a root of the function. Name an assumption that must hold concerning f in order for Newton's method (for root-finding) to converge quadratically to this root.

6. [**5 points**] (a) What is the form of the local truncation error for a finite-difference method for IVP's whose fixed time step is h and whose order is 4? (b) What is the form of the global error for this method (say, for integrating the IVP from 0 to 1)?

7. [**5 points**] Name a method for solving the equations arising in an implicit finite-difference method for an initial value problem (IVP).

8. [**5 points**] Hunter Rawlings, the President of Cornell, is taller than Professor Vavasis. Can you name any other person who is taller than Professor Vavasis? [Full credit will also be given if you can name any person shorter than Professor Vavasis.]

9. [**10 points**] Write down a 2×2 matrix such that plain Gaussian elimination (no pivoting) applied to the matrix would cause overflow. (Assume that the largest representable floating point number is 10^{308}.) Is Gaussian elimination with partial pivoting also prone to overflow on your 2×2 example?

10. [**10 points**] Discuss the tradeoffs of Runge-Kutta versus Adams-Bashforth methods for integrating an IVP.

11. **[15 points]** Let f, g be two cubic spline functions defined over the interval $[0, 1]$. Suppose the breakpoints are different. In particular, suppose the breakpoints (knots) of f are at $0, .1, .2, \ldots, .9, 1$ whereas the breakpoints of g are at $0, 1/3, 2/3, 1$. Is it true that $f + g$ is necessarily a cubic spline function? Explain why or why not.

12. **[15 points]** Let A be a $(2n) \times n$ matrix of the form

$$A = \begin{pmatrix} B \\ U \end{pmatrix}$$

where B is an $n \times n$ arbitrary matrix and U is an $n \times n$ upper triangular matrix. How many flops, accurate to the leading term, are required to factor A as QR using Givens' algorithm? Here Q is a $(2n) \times (2n)$ orthogonal matrix represented implicitly as a product of Givens rotations and R is an upper triangular matrix. Be sure to take advantage of structure so as not to carry out unnecessary operations. Explain your answer.

13. **[15 points]** The LDM factorization of an $n \times n$ matrix A is defined to be a factorization $A = LDM$, where L is lower triangular with 1's on the diagonal, D is a diagonal matrix, and M is an upper triangular matrix also with 1's on the diagonal. Propose an algorithm for computing this factorization of A. [Hint: consider the derivation of Cholesky presented in lecture.] If your algorithm could fail in some cases, please identify those cases.

14. **[15 points]** Consider the IVP in the special form $dy/dt = a(t)y + b(t)$, $y(0) = y_0$. Here, $a(t)$, $b(t)$ and $y(t)$ are all scalar functions of t. Write out the Backward Euler (BE) method for this problem. Derive an explicit formula for y^{k+1} for BE. (Even though BE is an implicit method, there is an explicit formula in this special case.) Does the explicit formula ever have a problem with divide-by-zero? Explain.

15. **[15 points]** Let n be even, let A be an $(n/2) \times n$ matrix, let $\mathbf{b}$ be an $n/2$-vector, and let $g : \mathbf{R}^n \to \mathbf{R}^n$ be a nonlinear function such that $[g_1(\mathbf{x}); \ldots; g_{n/2}(\mathbf{x})]$ (i.e., the vector made up of the half of the entries of $g(\mathbf{x})$) is always exactly equal to $A\mathbf{x} - \mathbf{b}$. If we apply Newton's method to solve $g(\mathbf{x}) = \mathbf{0}$ starting from $\mathbf{x}^{(0)}$ yielding iterates $\mathbf{x}^{(1)}$, $\mathbf{x}^{(2)}$, etc., what special property will the first $n/2$ entries of the vectors $g(\mathbf{x}^{(1)}), g(\mathbf{x}^{(2)})$ and so on have? Explain your answer. You can make assumptions concerning linear independence and rank, but please state your assumptions.