

Slide 1

**2 question** (1)

**looping questions**

I really don't understand why we are talking about loops in this fashion (recitation 3). It is totally unintuitive and counterintuitive.

**1 the instructor's answer**, where instructors collectively construct a single answer

I, Giles, have 16 years of programming experience.  
You have perhaps 1. I don't think your knowledge of programming is enough for you to make a decision like this.

We teach this material because we KNOW that it is the appropriate way to think about loops.

Do you play golf? A beginner plays golf, and misses the ball and slices it all over the place.  
He takes lessons from a pro. The pro tells him how to swing, and the beginner complains, "this doesn't  
bringtght at all, it's not intuitive". The pro says, "if you want to learn, do what I say. After a while, the  
beginner says, "hey, it does feel better and I'm no longer slicing so much".

Wait, you are the beginner and my job is to teach you professional ways of programming.


Tomorrow, I'll develop function evaluate from A2 for you, you will be amazed.

Later, we will develop lots of searching and sorting algorithms -- using loop invariants. The use of loop invariants  
will make it 500 times easier.

So put your doubts aside and remember that you are taking this class because you want to learn.

David Giles

Credit: Prof. David Giles, former professor of CS 2112, on Piazza (professor of Ed)



Slide 2

**CS 2112**

Recitation 8  
Loop Invariants

October 22 / 23, 2024

Slide 3

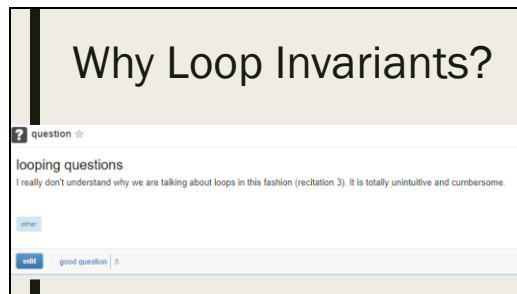
Agenda	Reminders
<ul style="list-style-type: none"><li>■ Loop Invariants</li><li>■ Bucket Sort</li><li>■ Binary Search</li><li>■ Exponentiation</li></ul>	<ul style="list-style-type: none"><li>• TA Evaluations</li><li>• A4 is a thing</li></ul>

Slide 4



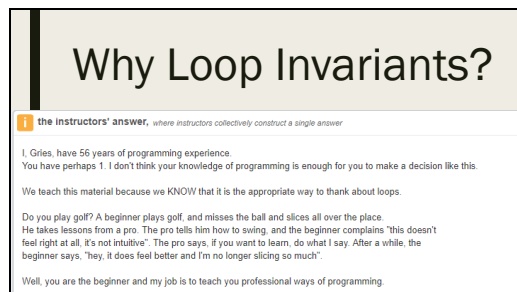
A brief review of loop invariants, which you saw in lecture

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The first question we inevitably get when talking about loop invariants is, "why do we study loop invariants"

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Prof. David Gries, former CS 2110 professor, had a particularly famous response to this question

[illegible]

# Why?


```
for ( int i = 0; i < 10; i++ ) {  
    System.out.println(i);  
}
```

[illegible]

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## Simplify

```
for ( <initialize>, <guard>, <increment> ) {  
    // Do Something  
}
```



```
<initialize>  
while ( <guard> ) {  
    // Do Something  
    <increment>  
}
```

Since all for loops can be written as a while loop, for simplicity, we will only be looking at while loops today.

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## Action Plan

```
<initialize>  
while ( <guard> ) {  
    // Do Something  
    <increment>  
}
```

**Goal**

**Generalized Statement**  
True after any number of iterations

The biggest reason loops are so hard to reason about is that they can run any number of times. So, in the proud tradition of students everywhere, when a problem is too hard, we'll give up and tackle an easier problem. The key idea is to come up with some generalized statement that is true after any number of iterations of the loop. By doing so, we can then reason without needing to worry about how many times the loop runs, sidestepping the hard part.

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## Action Plan

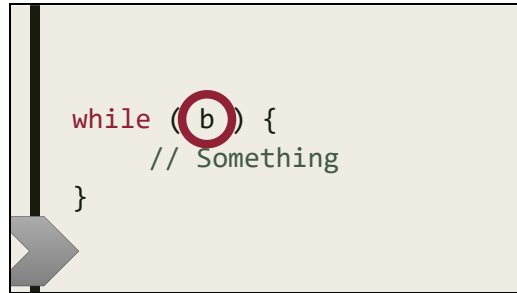
```
<initialize>  
while ( <guard> ) {  
    // Do Something  
    <increment>  
}
```

**Goal**

**Invariant**  
True after any number of iterations

You have probably guessed by now that the general statement is what we call the "Loop Invariant"

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When we exit a loop, the thing we know for a fact is that the loop guard (“b” in this case) must be false, or else the loop would still be running.

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### Action Plan

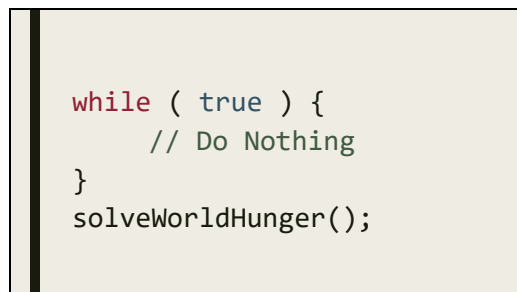
```
<initialize>  
while ( <guard> ) {  
    // Do Something  
    <increment>  
}
```

Invariant + !<guard>  $\Rightarrow$  Goal

**Invariant**  
True after any number of iterations

Therefore, our goal is to show that the loop invariant together with the negation of the loop guard proves the goal we want our loop to fulfill.

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The other caveat is that a loop which doesn't end will never exit, and thus our loop guard will never be false.

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## Action Plan

```

<initialize>
while ( <guard> ) {
  // Do Something
  <increment>
}

```

**Invariant**  
True after any number of iterations

Invariant + !<guard>  $\Rightarrow$   
**Goal**  
(as long as the loop actually ends)

Therefore, a complete proof also needs to show that the loop actually ends.

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## Invariant

True after any number of iterations

**$n = 0$**   
True after zero iterations (after initialization)

**$n + 1$**   
True after any particular iteration if true at beginning

To show that the invariant is true after any number of iterations, we first must show it is true for the smallest number of iterations a loop can make, which is zero (NOT one!). Then, we can show that it's true for any  $n+1$  iterations if it was true for  $n$  iterations. This is called induction.

(in fact, a loop invariant proof is exactly a proof by induction, where the loop invariant is our inductive hypothesis)

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## Action Plan

```

<initialize>
while ( <guard> ) {
  // Do Something
  <increment>
}

```

**Invariant + !<guard>  $\Rightarrow$  Goal**  
(as long as the loop actually ends)

True after zero iterations (after initialization)

True after any particular iteration if true at beginning

Thus here are the four steps we need to prove to complete our proof of the loop's correctness

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## Action Plan

```
<initialize>
while ( <guard> ) {
  // Do Something
  <increment>
}
```

Establishment (true after zero initialization)

Preservation (True after any if true at beginning)

Postcondition (Invariant + <guard>  $\Rightarrow$  goal)

Termination (as long as the loop actually ends)

We can assign fancy names to these steps

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## Loop Invariant Proof

- Establishment
- Postcondition
- Preservation
- Termination

And hopefully you recognize these names as the four parts you were taught in lecture.

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## SIMPLE EXAMPLE

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**Task**

Given an array with two types of elements,  
sort the array.

Here's an actual interview question I got from Microsoft when applying for my internship there.

I'm going to show you a foolproof four-step plan to write any loop correctly.

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**Step 1**

Draw the state at the start of the loop

b 

?
---

I'm using pictures here instead of words for intuition. CS 2110 used to teach loop invariants exclusively with pictures, and while we do expect 2112 students to be able to formalize their invariants in writing, I do still find the pictures useful for organizing my thoughts.

In this picture, at the start of the loop, nothing is sorted yet, so everything is marked with a "?", indicating that the invariant makes no promises about its contents (other than the obvious stuff, like that there's two types of things in here)

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**Step 2**

Draw the state at the end of the loop

b 

?
---

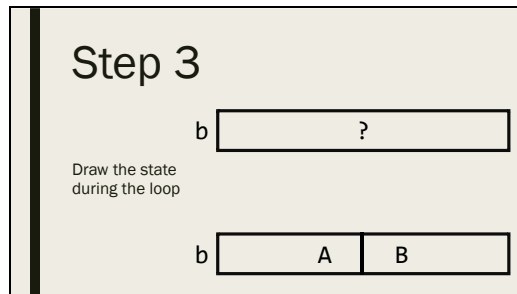
b 

A	B
---	---

At the end of the loop, the two types of things should be properly sorted (marked by "A" and "B").

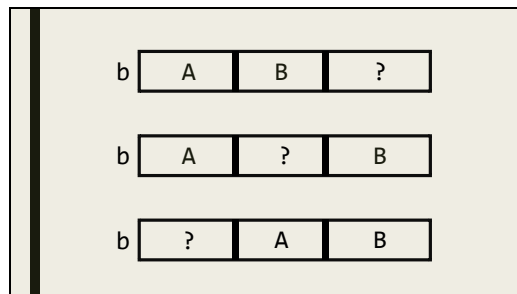


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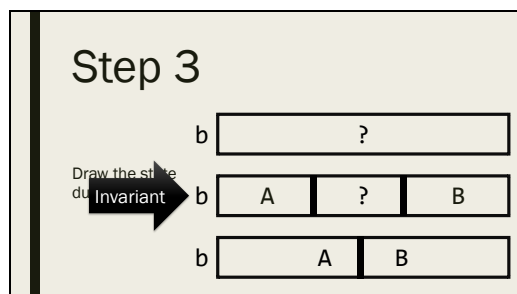
Now we need to visualize how the loop should transition from the start to the end.

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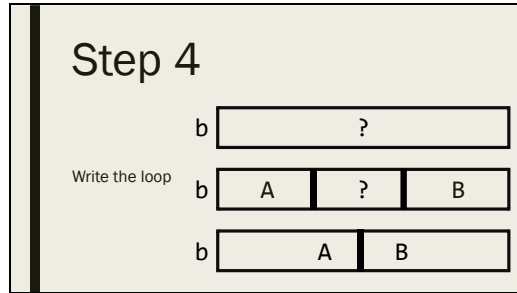
There's usually multiple ways to make that transition. In these three examples, you can see the sorted section growing in from either end of the array. The point is, pick one.

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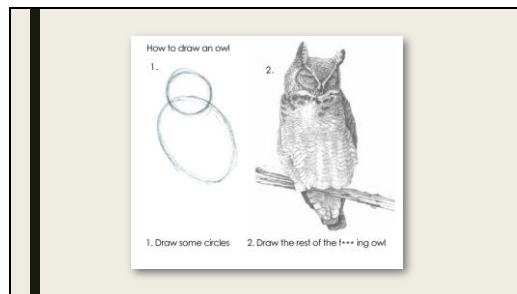
Unsurprisingly, this intermediate state you picked is going to form your loop invariant.

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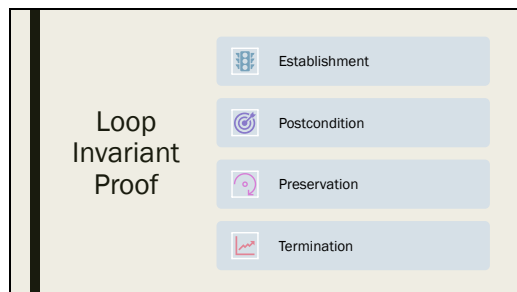
Final step is to write the loop

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Admittedly, this final step does not sound useful. But I promise that we're onto something here...

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Because the four steps of the loop invariant proof can now guide our loop writing process.

## Slide 31

Establishment

Postcondition

Preservation

Termination

```
int i = 0;
int j = b.length;
while ( i != j ) {
    if (b[i] == A) {
        i++;
    }
    j--;
    swap(b[i], b[j]);
}
```

b

i

j

A

?

B

$\forall x, x < i \quad b[x] == A$

$\forall x, x \geq j \quad b[x] == B$

Establishment forces `i` and `j` to be certain values at the start. The post condition lets us reason about what must be true at the end of the loop (`i == j`), so we negate that to form the loop guard. Then, termination tells us how to make progress (`i++` or `j--`) and then preservation forces the rest of the loop to be correct.

This was the exact loop I wrote for my interviewer. I got the job.

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Task

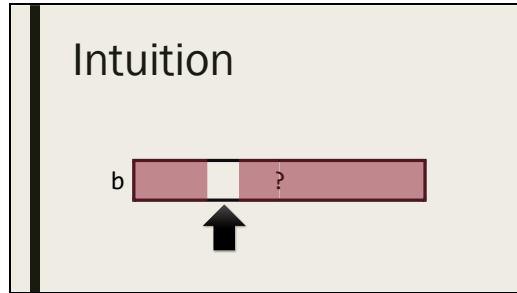
Given an array with <sup>three</sup>~~two~~ types of elements, sort the array.

Now you try. This was the followup question my interviewer asked me. Same setup, but there's three types of elements now.

## Slide 33

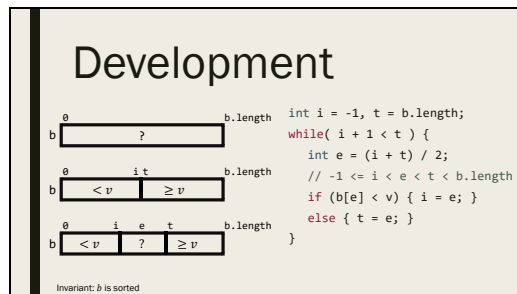
BINARY SEARCH

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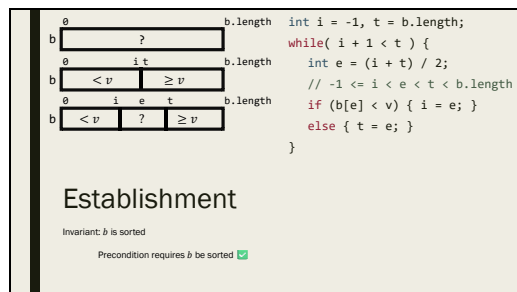
You saw this in a previous discussion; binary search finds things in a sorted array quickly by jumping to the middle of the remaining half at each step.

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We can use the same loop invariant formalism to write the loop. Next, we'll practice proving the loop's correctness.

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The invariant requires *b* be sorted, but that's also the precondition of the binary search, so trivially this is satisfied.

# Slide 37

Diagram 1: Array  $b$  from index 0 to  $b.length$ . Element  $e$  is at index  $i$ . Elements before  $e$  are  $< v$ , elements after are  $\geq v$ .

Diagram 2: Array  $b$  from index 0 to  $b.length$ . Element  $e$  is at index  $i$ . Elements before  $e$  are  $< v$ , elements after are  $\geq v$ .

Invariant:  $b$  is sorted

```

int i = -1, t = b.length;
while( i + 1 < t ) {
    int e = (i + t) / 2;
    // -1 <= i < e < t < b.length
    if (b[e] < v) { i = e; }
    else { t = e; }
}

```

Establishment

Diagram 3: Array  $b$  from index 0 to  $b.length$ . Element  $e$  is at index  $i$ . Elements before  $e$  are  $< v$ , elements after are  $\geq v$ .

$-1 \leq i < t \leq b.length$

$i = -1$  ✓  
 $t = b.length$  ✓  
 $i < t$  ✓

Next, we have bounds on  $i$  and  $t$ , but these are also trivially true at the start of the loop.

# Slide 38

Diagram 1: Array  $b$  from index 0 to  $b.length$ . Element  $e$  is at index  $i$ . Elements before  $e$  are  $< v$ , elements after are  $\geq v$ .

Diagram 2: Array  $b$  from index 0 to  $b.length$ . Element  $e$  is at index  $i$ . Elements before  $e$  are  $< v$ , elements after are  $\geq v$ .

Invariant:  $b$  is sorted

```

int i = -1, t = b.length;
while( i + 1 < t ) {
    int e = (i + t) / 2;
    // -1 <= i < e < t < b.length
    if (b[e] < v) { i = e; }
    else { t = e; }
}

```

Establishment ✓

Diagram 3: Array  $b$  from index 0 to  $b.length$ . Element  $e$  is at index  $i$ . Elements before  $e$  are  $< v$ , elements after are  $\geq v$ .

$\forall x, 0 \leq x \leq i \quad b[x] < v$   
 $\forall x, t \leq x < b.length \quad b[x] \geq v$

$\{x : 0 \leq x \leq -1\} = \emptyset$   
 $\{x : b.length \leq x < b.length\} = \emptyset$

Most importantly, our invariant guarantees certain chunks of the array are smaller or larger than  $v$ , the target value. Luckily, at the start, we've chosen values such that these chunks of the array are empty, meaning it's trivially true.

Establishment tends to be the easy part to prove, in general.

# Slide 39

Diagram 1: Array  $b$  from index 0 to  $b.length$ . Element  $e$  is at index  $i$ . Elements before  $e$  are  $< v$ , elements after are  $\geq v$ .

Diagram 2: Array  $b$  from index 0 to  $b.length$ . Element  $e$  is at index  $i$ . Elements before  $e$  are  $< v$ , elements after are  $\geq v$ .

Diagram 3: Array  $b$  from index 0 to  $b.length$ . Element  $e$  is at index  $i$ . Elements before  $e$  are  $< v$ , elements after are  $\geq v$ .

Invariant:  $b$  is sorted

```

int i = -1, t = b.length;
while( i + 1 < t ) {
    int e = (i + t) / 2;
    // -1 <= i < e < t < b.length
    if (b[e] < v) { i = e; }
    else { t = e; }
}

```

Postcondition

$!( \text{ i + 1 < t } ) \Rightarrow i + 1 \geq t$

Next, the postcondition tells us the loop guard must be false, which means we negate the part inside the loop guard.

# Slide 40

0

b

?

b.length

0

i

t

b.length

b

< v

?

≥ v

e

```

int i = -1, t = b.length;
while( i + 1 < t ) {
    int e = (i + t) / 2;
    // -1 <= i < e < t < b.length
    if (b[e] < v) { i = e; }
    else { t = e; }
}

```

Postcondition

✓

$-1 \leq i < t \leq b.length$   
 $i + 1 = t$

$i + 1 \geq t$   
 Invariant:  $b$  is sorted

0

i

t

b.length

b

< v

?

≥ v

$\forall x, 0 \leq x \leq i \quad b[x] < v$   
 $\forall x, t \leq x < b.length \quad b[x] \geq v$   
 $b[t] = v \text{ if } v \text{ in } b$

Using this fact, we can see that given our bounds requires  $i < t$ , yet  $i + 1 \geq t$ , we must conclude that  $i + 1 = t$ . In other words,  $i$  is the index of an element immediately before  $t$ . But then, our invariant tells us  $b$  is sorted, and the element at index  $i$  is  $< v$ , while the element at index  $t$  is  $\geq v$ . If  $i$  comes immediately before  $t$ , then the element at  $t$  must be  $v$ , since if  $v$  can't be earlier in the array since  $i$  must be less than  $v$ , and it can't be later since the array is sorted. This means we've found the position of  $v$ .

# Slide 41

0

b

?

b.length

0

i

t

b.length

b

< v

?

≥ v

e

```

int i = -1, t = b.length;
while( i + 1 < t ) {
    int e = (i + t) / 2;
    // -1 <= i < e < t < b.length
    if (b[e] < v) { i = e; }
    else { t = e; }
}

```

Invariant:  $b$  is sorted

✓

Preservation

✓

$-1 \leq i < t \leq b.length$   
 $\forall x, 0 \leq x \leq i \quad b[x] < v$   
 $\forall x, t \leq x < b.length \quad b[x] \geq v$

0

i

e

t

b.length

b

< v

?

≥ v

There's two cases inside the loop we need to examine for preservation. If  $b[e] < v$ , we see that we lift  $i$  up to  $e$ . Since  $e < t$ , the bounds are still correct. Since  $b[e] < v$  and  $b$  is sorted, all the things up to and including position  $e$  must be  $< v$ , which means we can safely lift  $i$  up to it. Finally, we don't modify  $t$  at all (nor do we modify the array), so the invariant about  $t$  and  $b$  being sorted are all maintained.

# Slide 42

0

b

?

b.length

0

i

t

b.length

b

< v

?

≥ v

e

```

int i = -1, t = b.length;
while( i + 1 < t ) {
    int e = (i + t) / 2;
    // -1 <= i < e < t < b.length
    if (b[e] < v) { i = e; }
    else { t = e; }
}

```

Invariant:  $b$  is sorted

✓

Preservation

✓

$-1 \leq i < t \leq b.length$   
 $\forall x, 0 \leq x \leq i \quad b[x] < v$   
 $\forall x, t \leq x < b.length \quad b[x] \geq v$

0

i

e

t

b.length

b

< v

?

≥ v

By the same logic, moving  $t$  down to  $e$  if  $b[e] \geq v$  is also safe (convince yourself this is true). Thus, we have proven preservation.

# Slide 43

0

b

?

b.length

0

b

i

t

b.length

0

b

< v

≥ v

b.length

0

b

i

e

t

b.length

0

b

< v

?

≥ v

b.length

Invariant: b is sorted

```

int i = -1, t = b.length;
while( i + 1 < t ) {
    int e = (i + t) / 2;
    // -1 <= i < e < t < b.length
    if (b[e] < v) { i = e; }
    else { t = e; }
}

```

Termination

✓

i < e < t

i = e

t = e

t - i gets smaller every loop

t - i = 1 terminates loop

Finally for termination, note that the quantity  $t - i$  must strictly decrease every loop since we either set  $i$  up to  $e$  or  $t$  down to  $e$ , and  $i < e < t$ . Therefore,  $t - i$ , an integer, decreases strictly every iteration, and when it equals 1, the loop ends. Since it starts at a value  $\geq 1$  and will never go below 1 (by the invariant), termination is guaranteed.

# Slide 44

0

b

?

b.length

0

b

i

t

b.length

0

b

< v

≥ v

b.length

0

b

i

e

t

b.length

0

b

< v

?

≥ v

b.length

Invariant: b is sorted

```

int i = -1, t = b.length;
while( i + 1 < t ) {
    int e = (i + t) / 2;
    // -1 <= i < e < t < b.length
    if (b[e] < v) { i = e; }
    else { t = e; }
}

```

Establishment

✓

Preservation

✓

Postcondition

✓

Termination

✓

This completes the loop invariant proof.

# Slide 45

EXPONENTIATION

## Slide 46

```
/** Returns: x^e
 * Requires: e ≥ 0
 * Performance: O(log e) */
static int pow(int x, int e) {
    int r = 1, b = x, y = e;
    // loop invariant: r·b^y = x^e and y ≥ 0
    while (y > 0) {
        if (y % 2 == 1) { r = r * b; }
        y = y / 2;
        b = b * b;
    }
    return r;
}
```

Now you try. Here's a loop that exponentiates. The invariant is given to you. Prove its correctness.

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