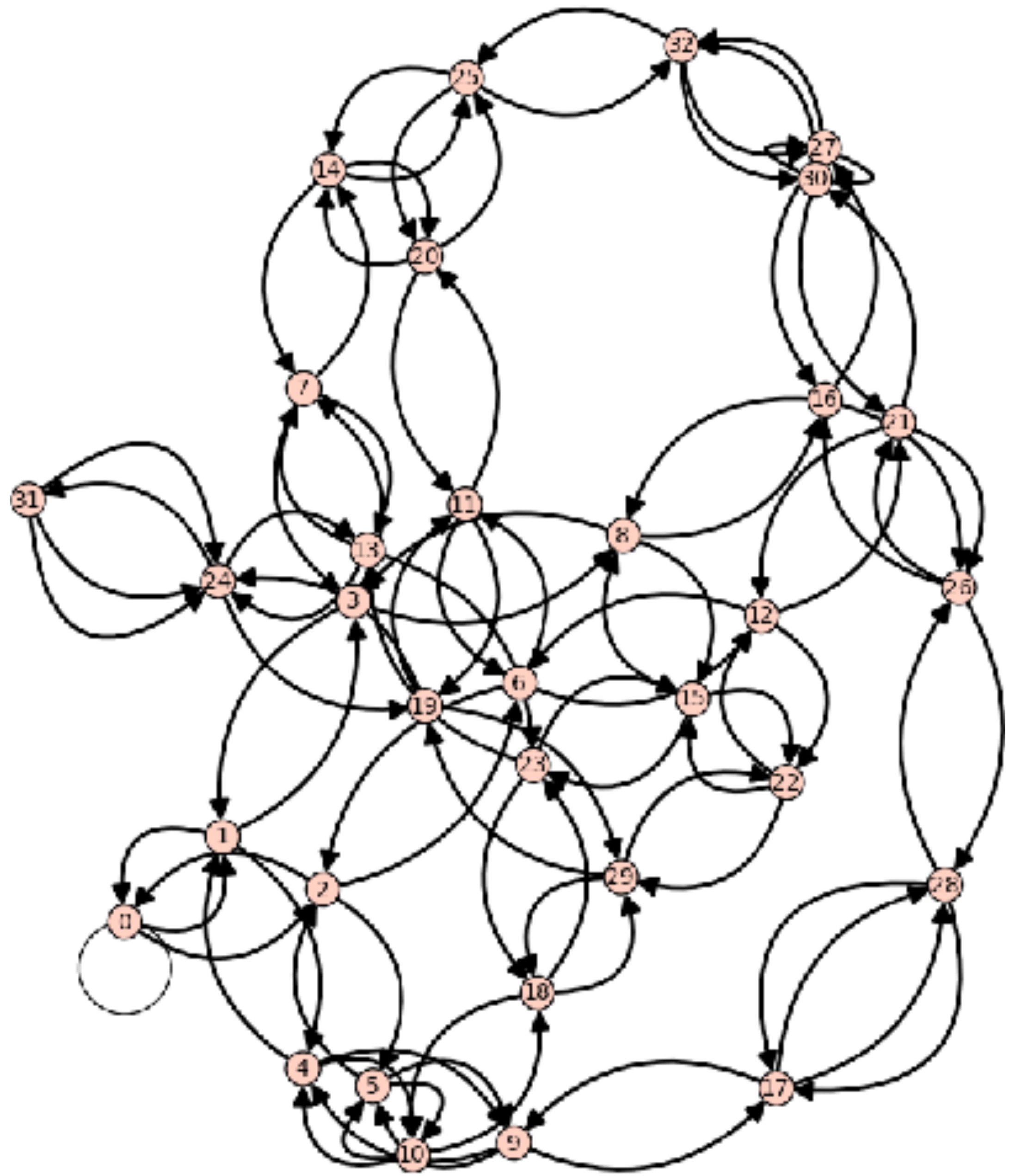
Minimum Spanning Trees

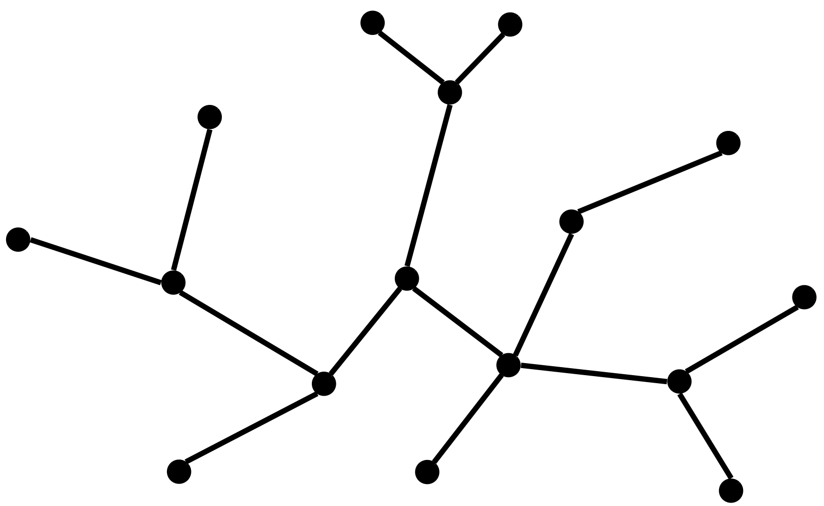
Recitation 13

CS 2112 Fall 2018



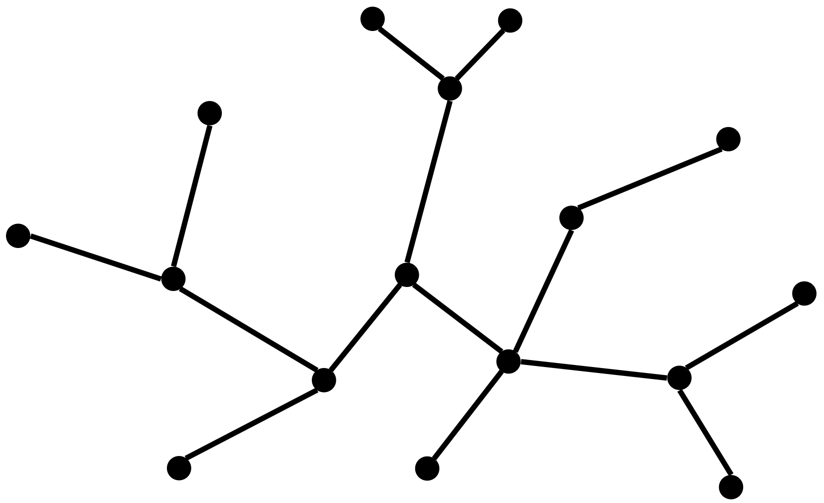
Undirected Trees

- An undirected graph is a *tree* if there is exactly one simple path between any pair of nodes



Undirected Trees

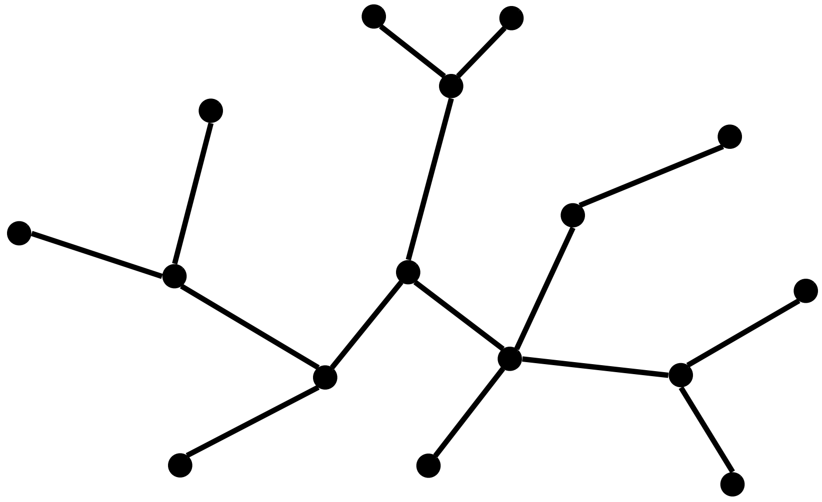
- Equivalently: an undirected graph is a *tree* if it is connected (there is a path between any pair of nodes) and acyclic



Facts About Trees

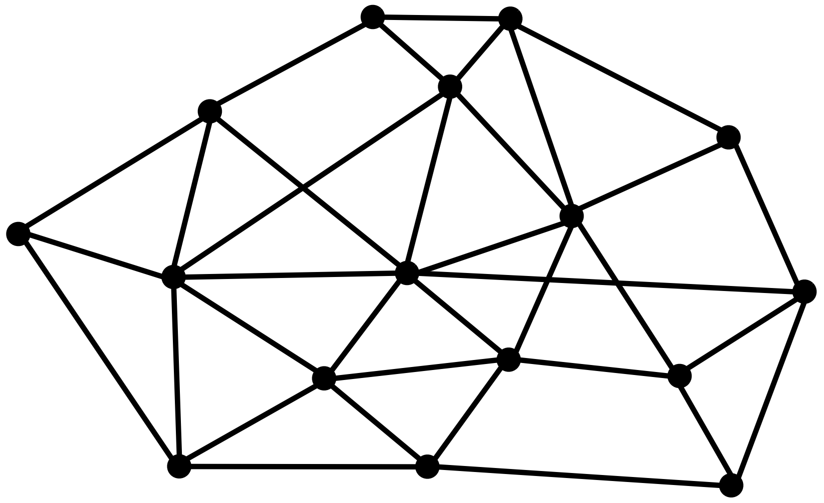
1. $|E| = |V| - 1$
2. connected
3. no cycles

Any two of these properties imply the third, and imply that the graph is a tree



Spanning Trees

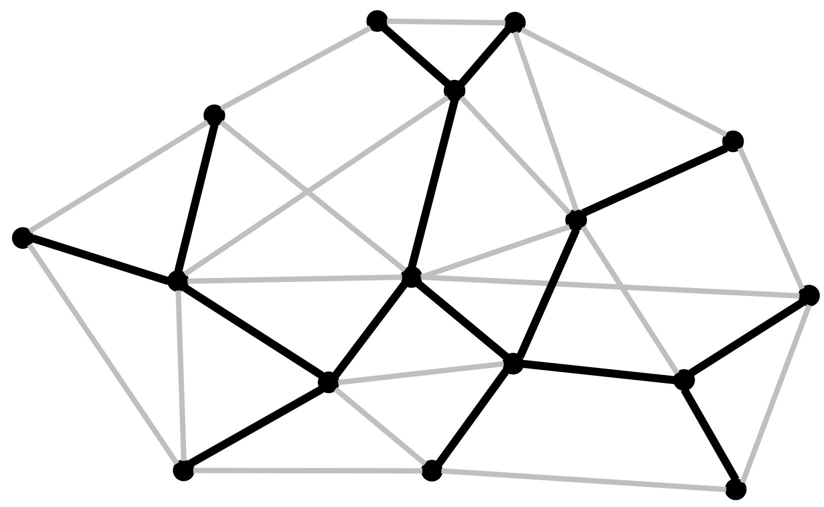
A *spanning tree* of a connected undirected graph (V, E) is a subgraph (V, E') that is a tree



Spanning Trees

A *spanning tree* of a connected undirected graph (V, E) is a subgraph (V, E') that is a tree

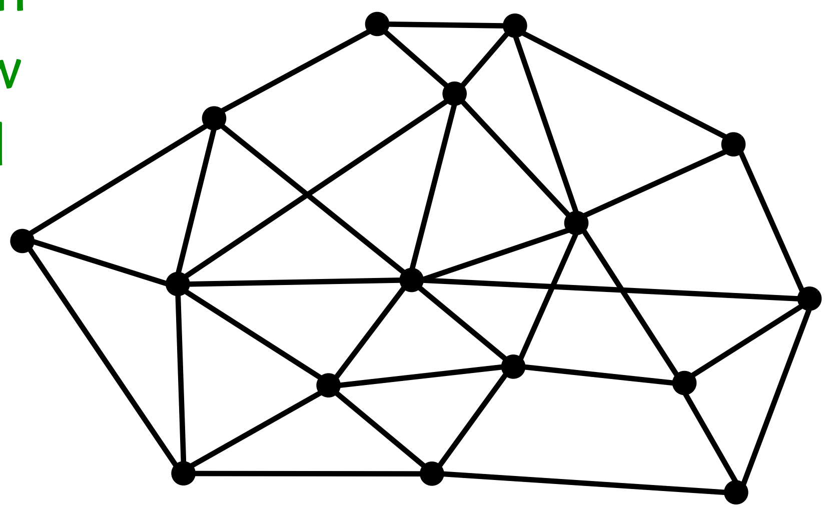
- Same set of vertices V
- E' ⊆ E
- (V, E') is a tree



Finding a Spanning Tree

A subtractive method

- Start with the whole graph – it is connected
- If there is a cycle, pick an edge on the cycle, throw it out – the graph is still connected (why?)
- Repeat until no more cycles



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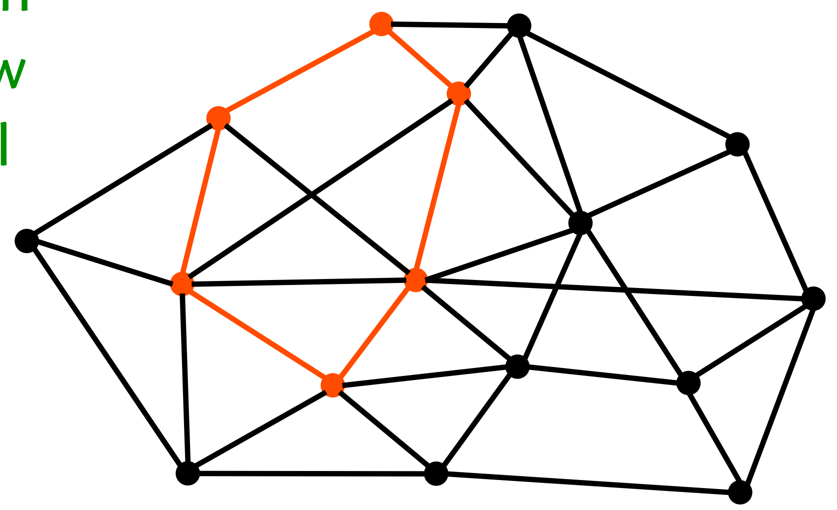

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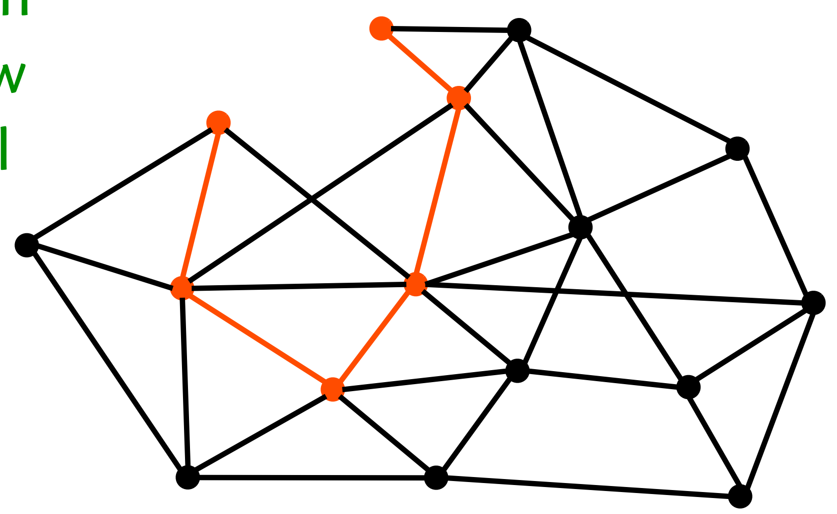
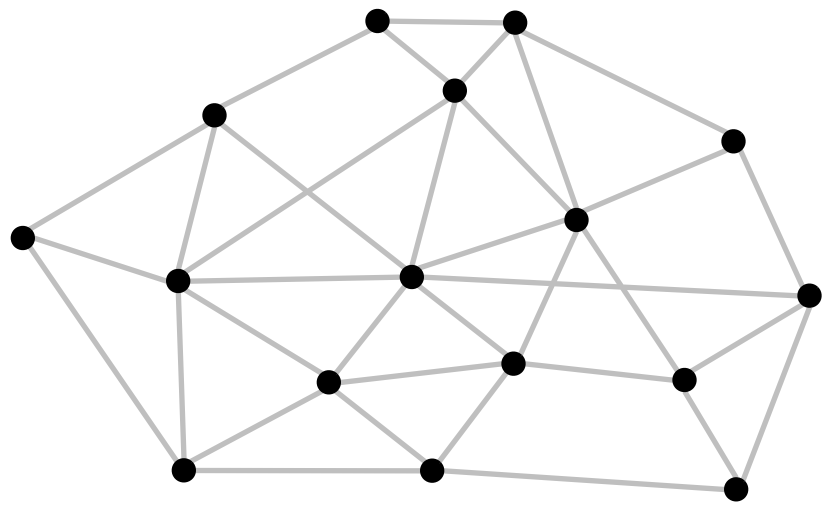


Finding a Spanning Tree

An additive method

- Start with no edges – there are no cycles
- If more than one connected component, insert an edge between them – still no cycles (why?)
- Repeat until only one component



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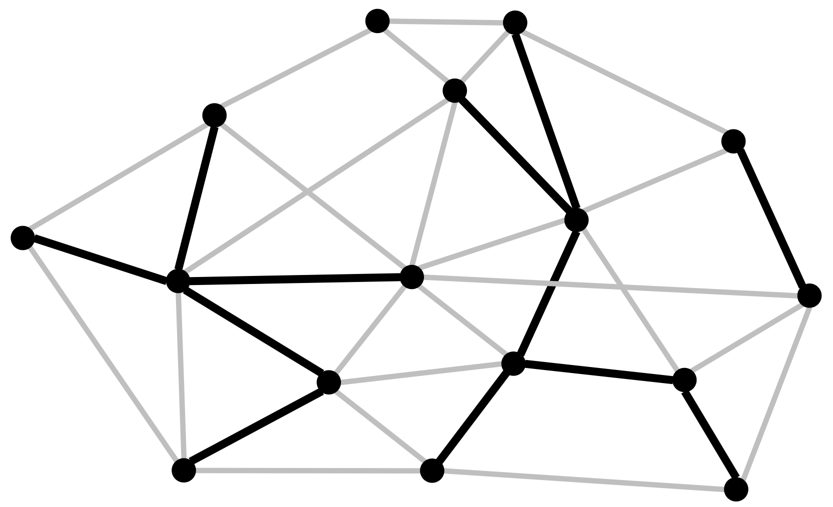

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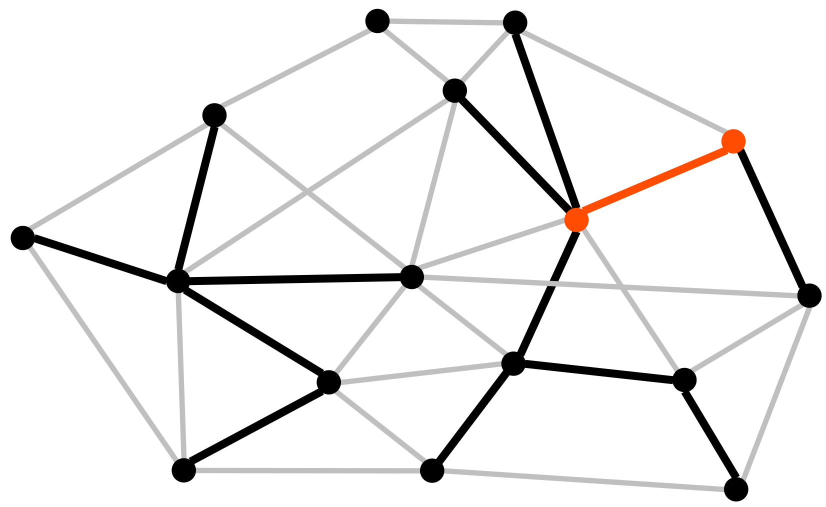

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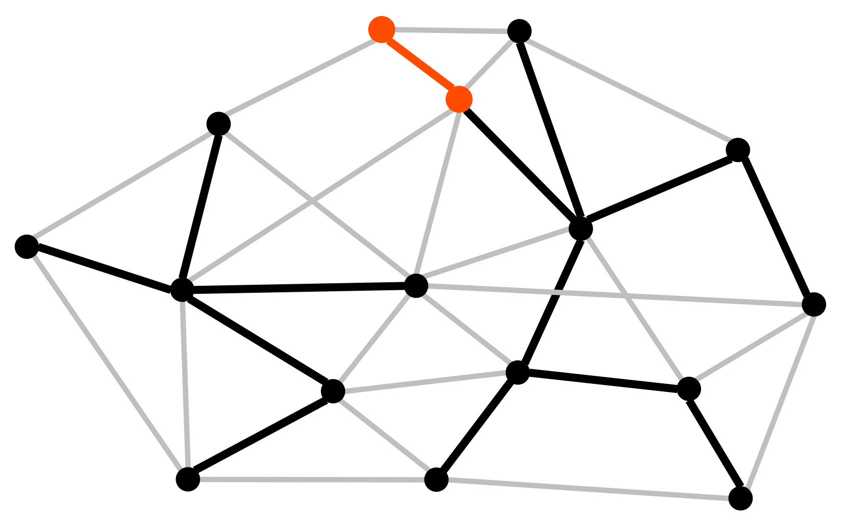

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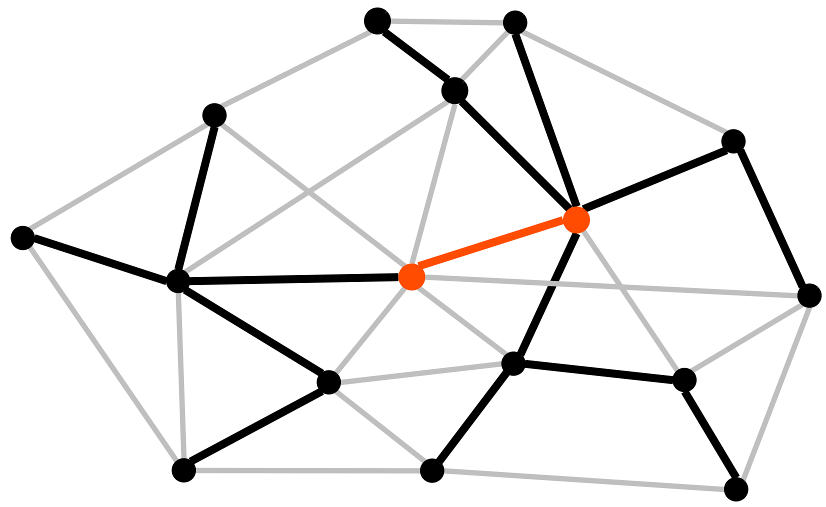

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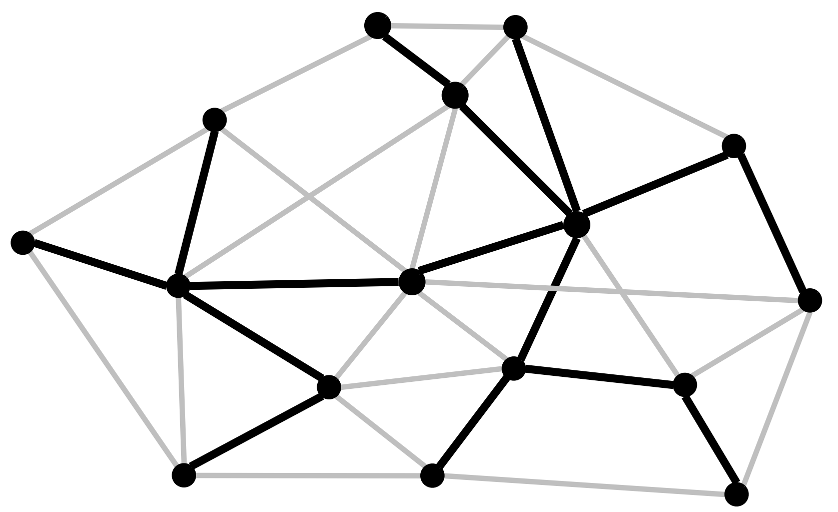
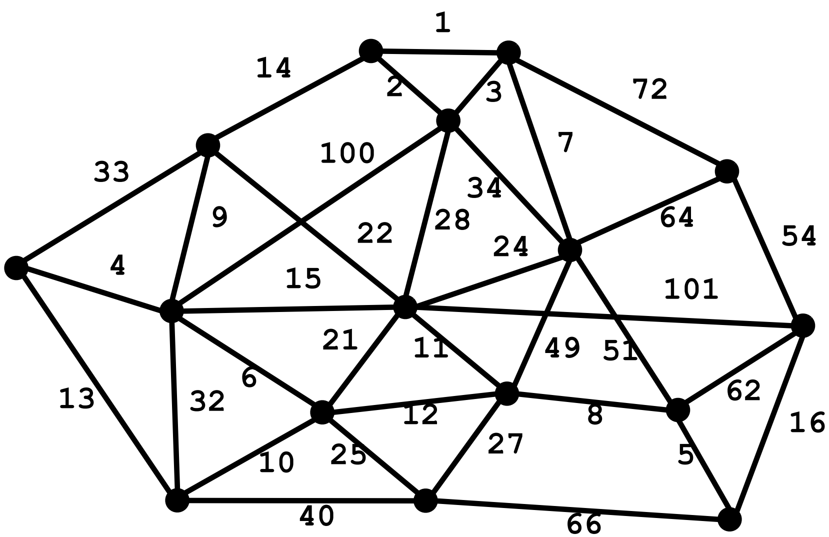


Minimum Spanning Trees

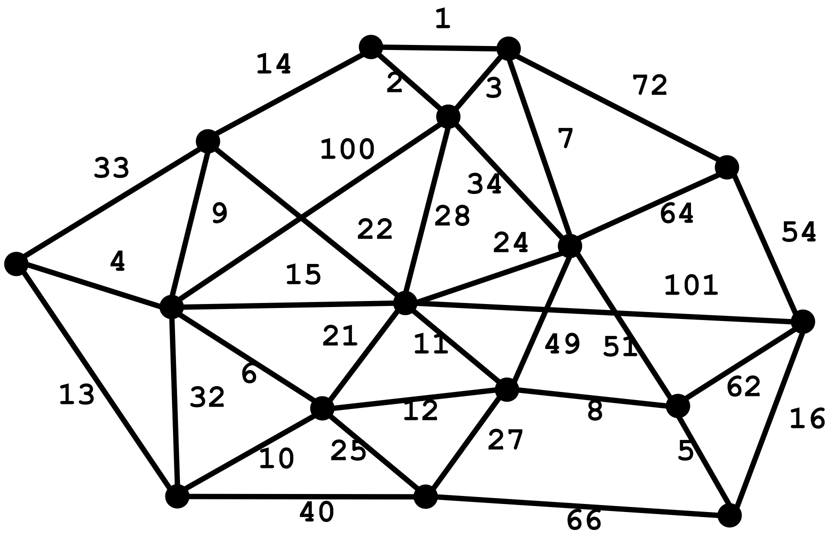
- Suppose edges are weighted, and we want a spanning tree of *minimum cost* (sum of edge weights)

- Useful in network routing & other applications



3 Greedy Algorithms

A. Find a max weight edge – if it is on a cycle, throw it out, otherwise keep it



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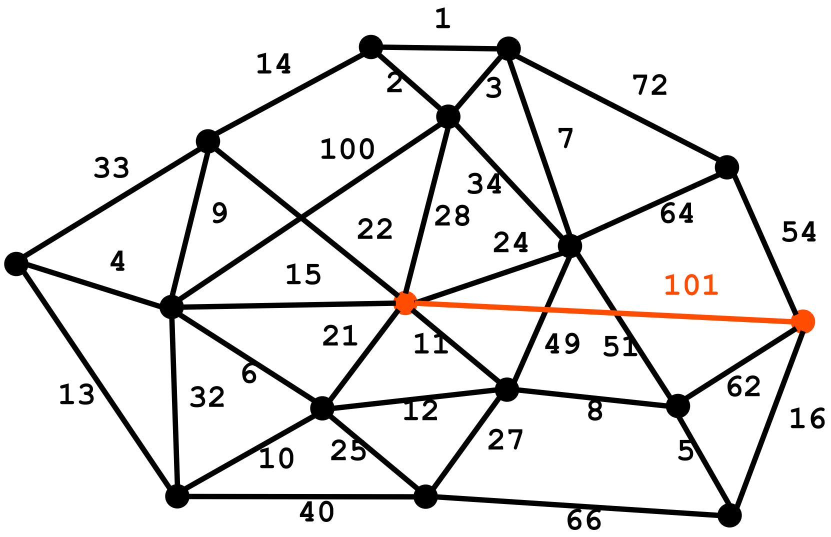

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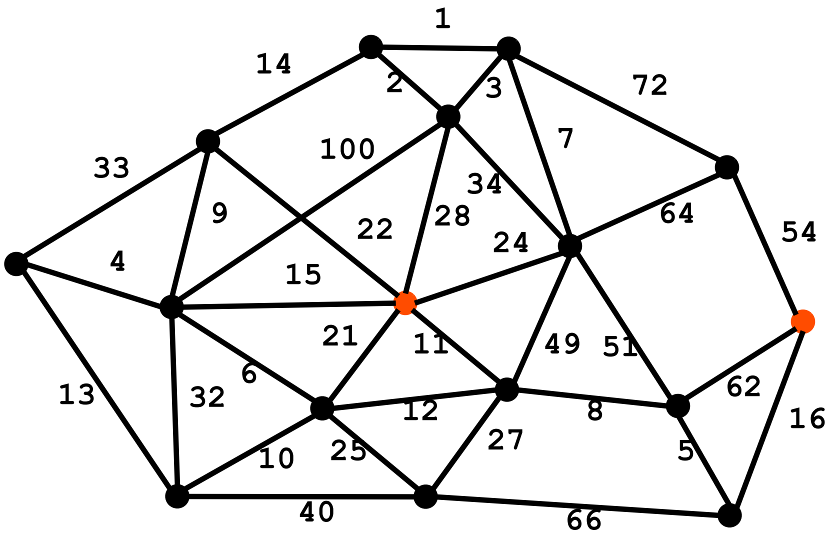

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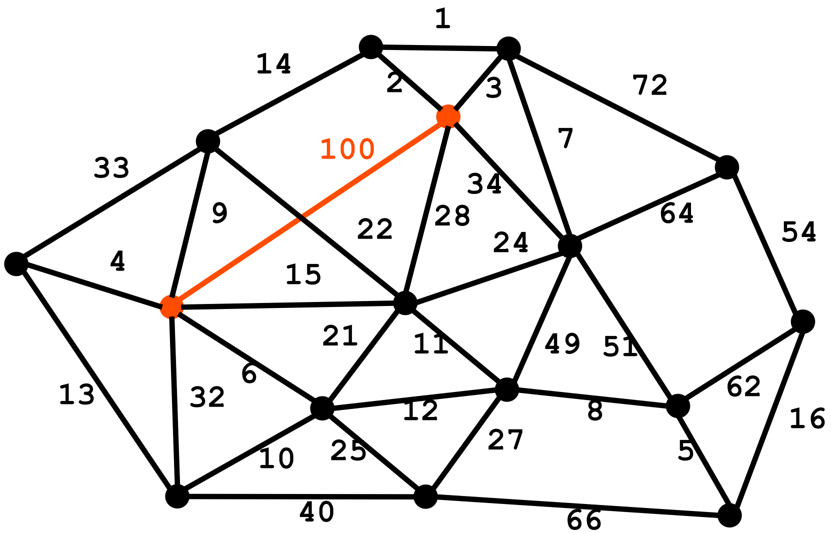

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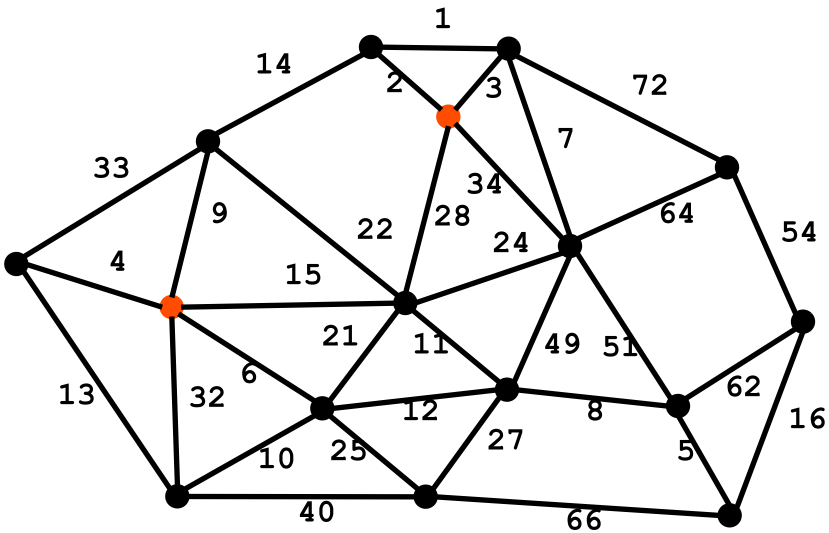

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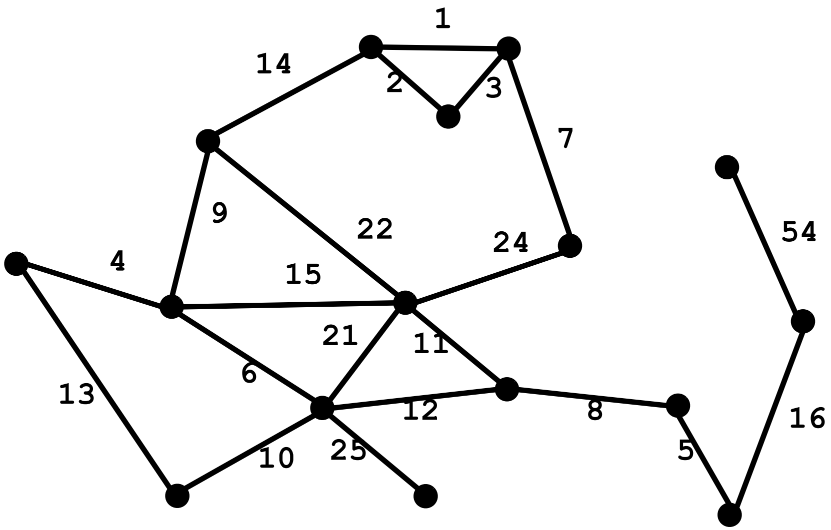

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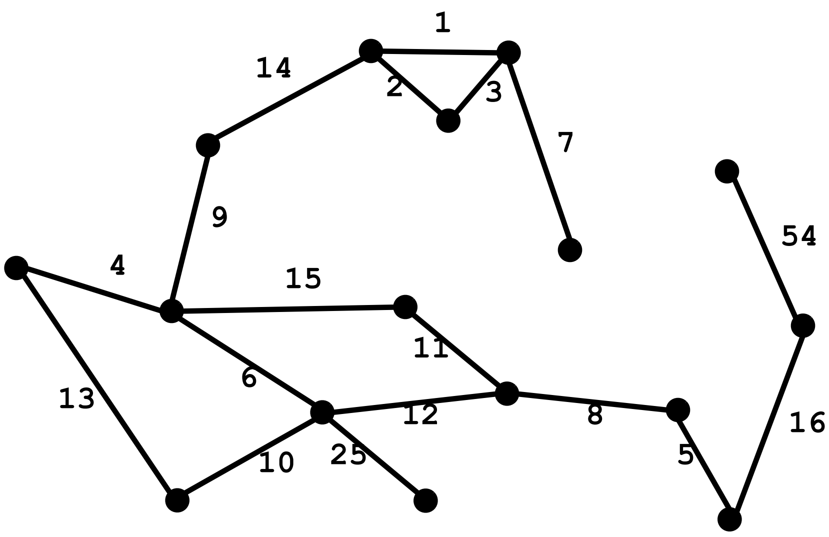

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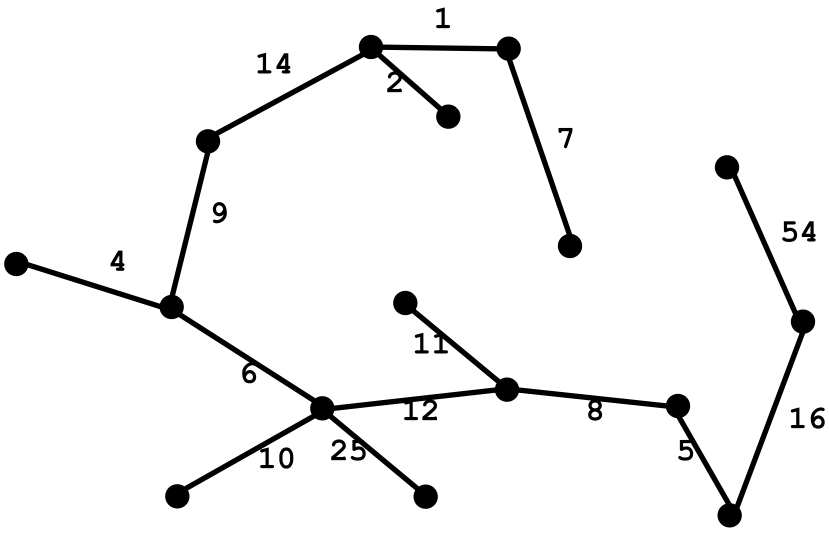
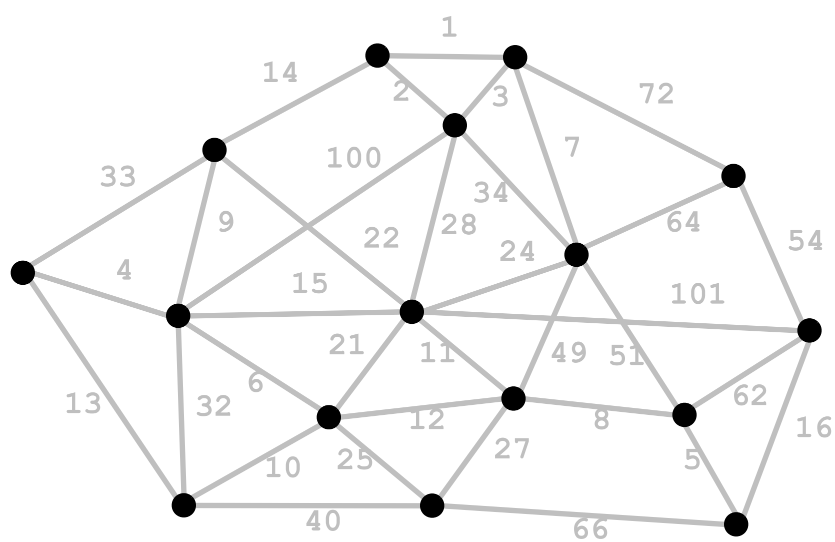


3 Greedy Algorithms

B. Find a min weight edge – if it forms a cycle with edges already taken, throw it out, otherwise keep it

Kruskal's algorithm



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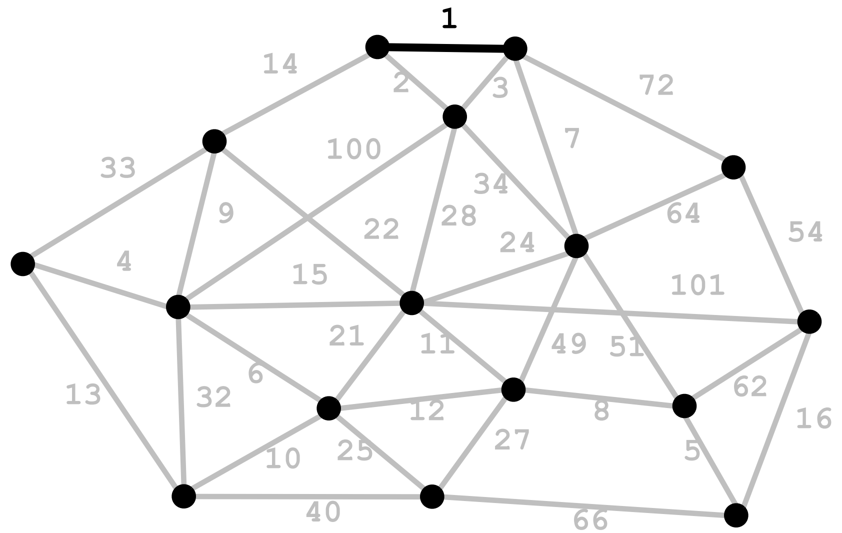

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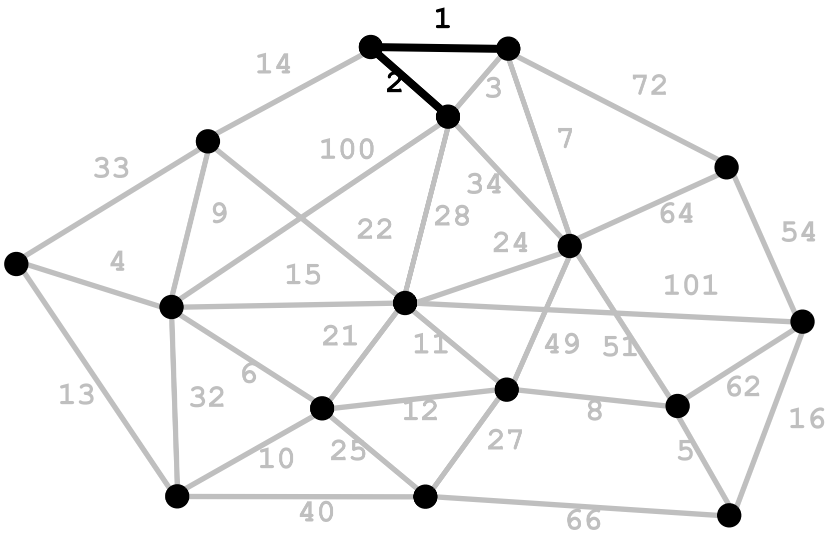

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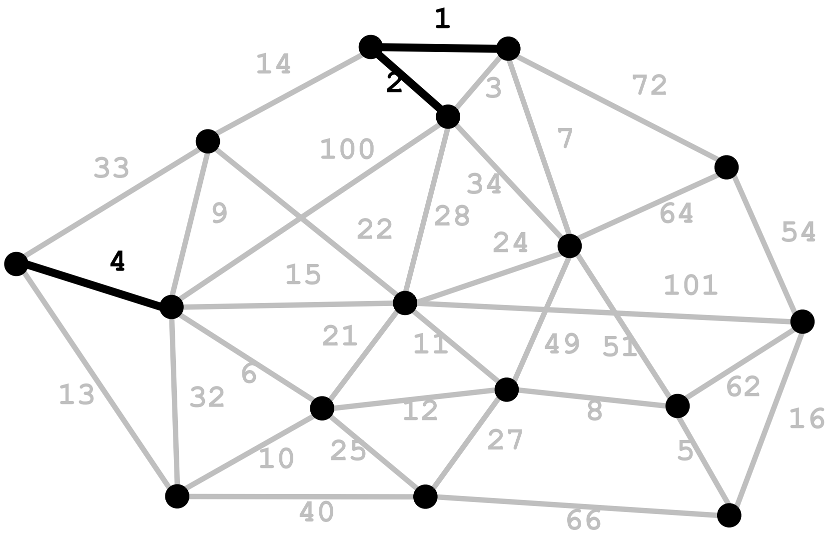

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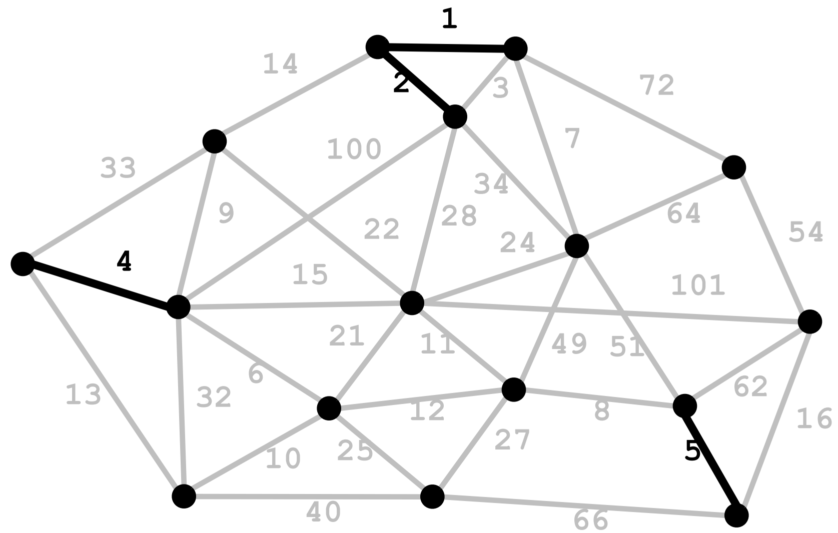

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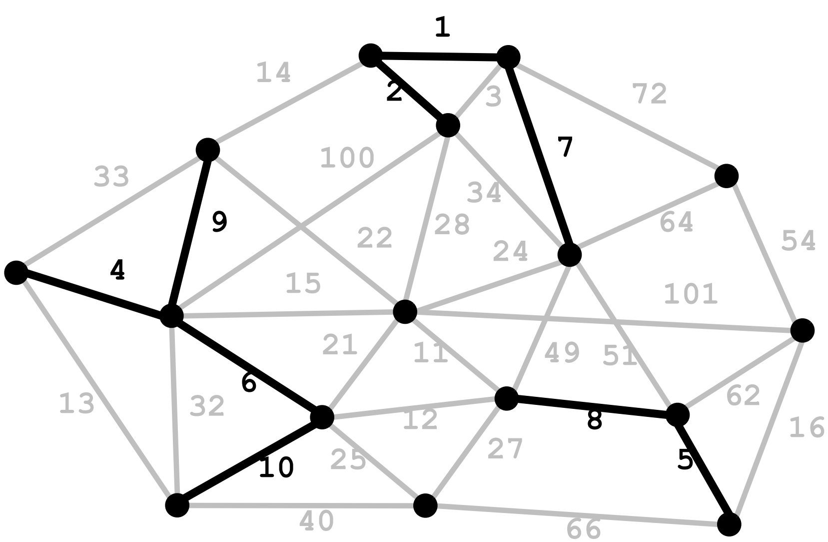

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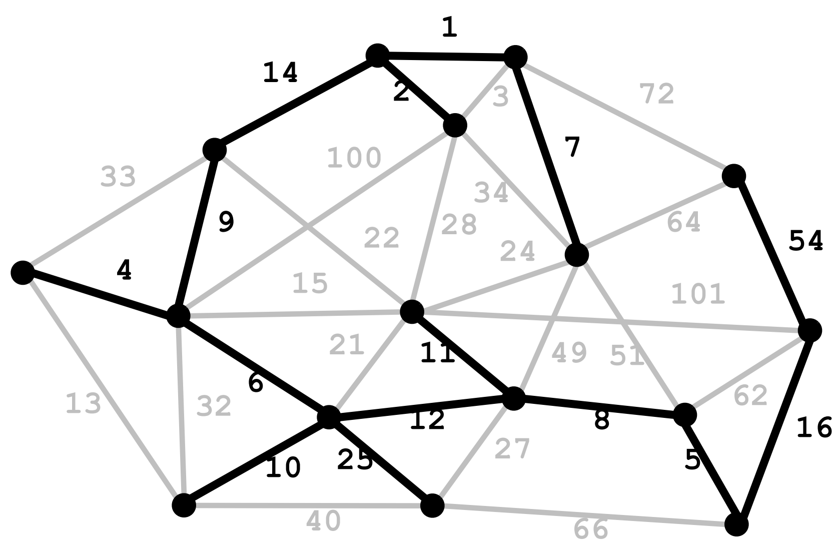
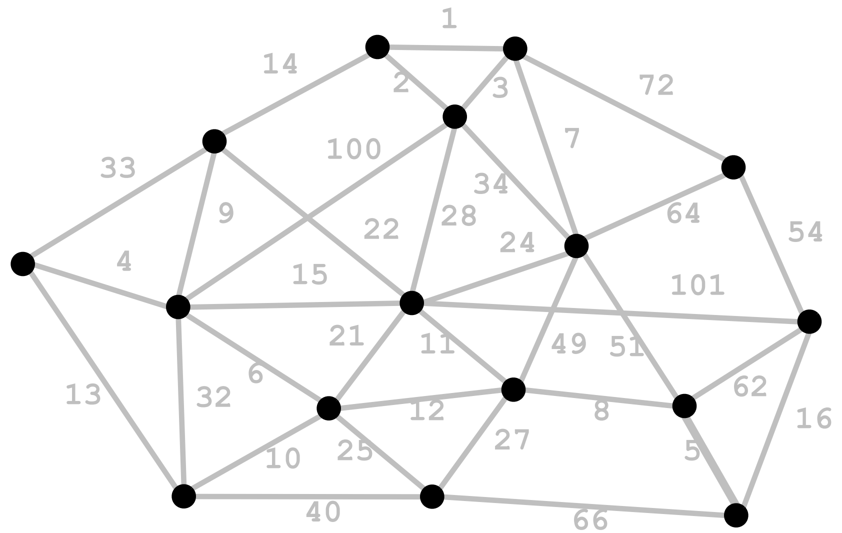


3 Greedy Algorithms

C. Start with any vertex, add min weight edge extending that connected component that does not form a cycle

Prim's algorithm

(reminiscent of Dijkstra's algorithm)



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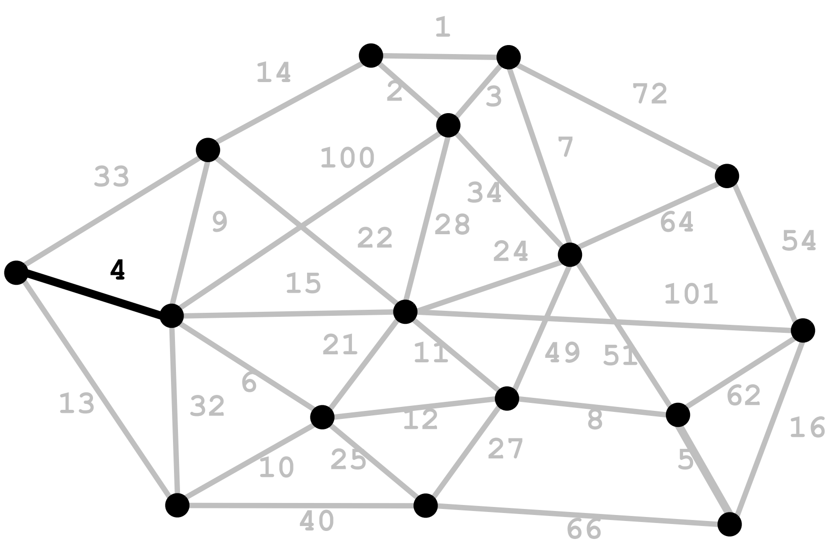

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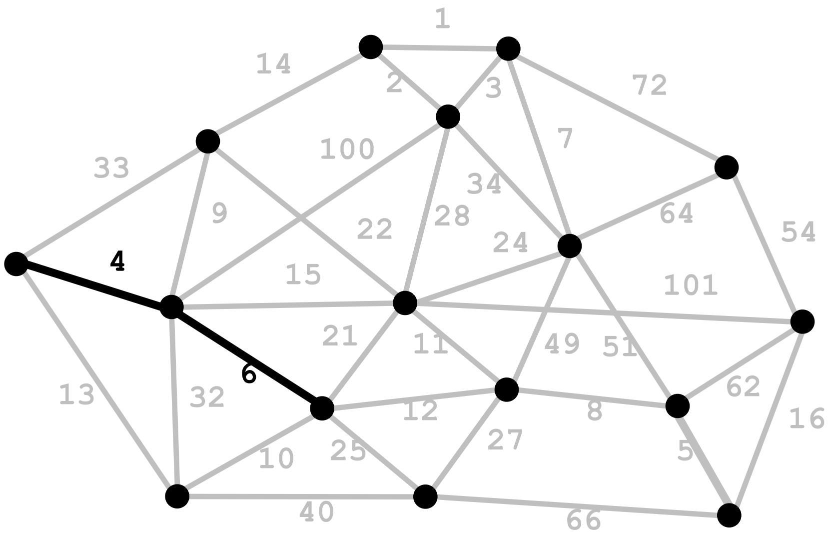

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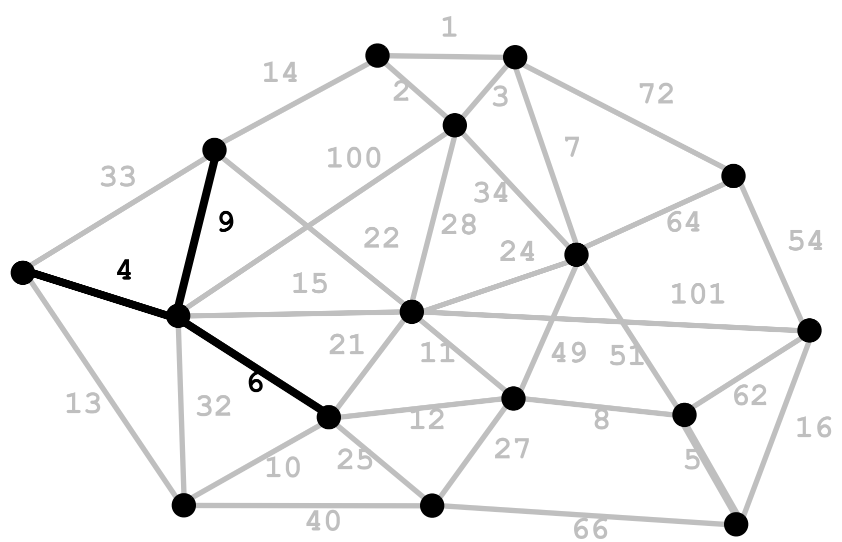

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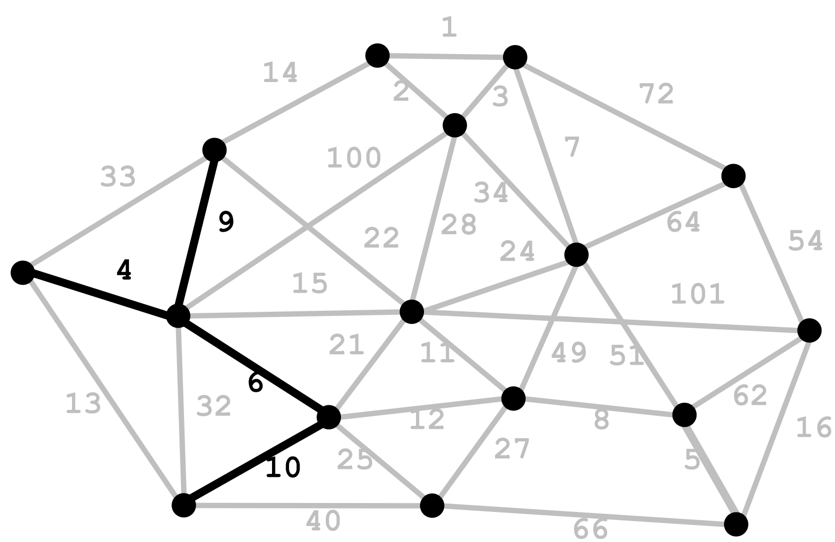

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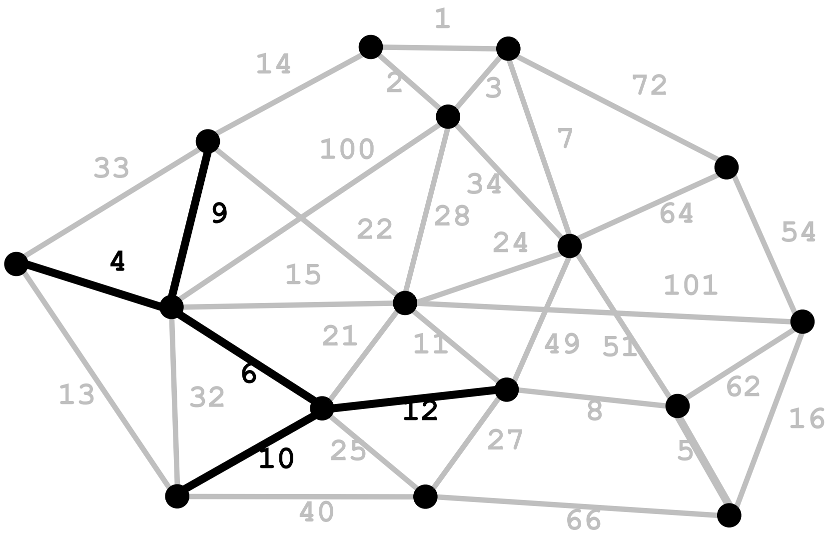

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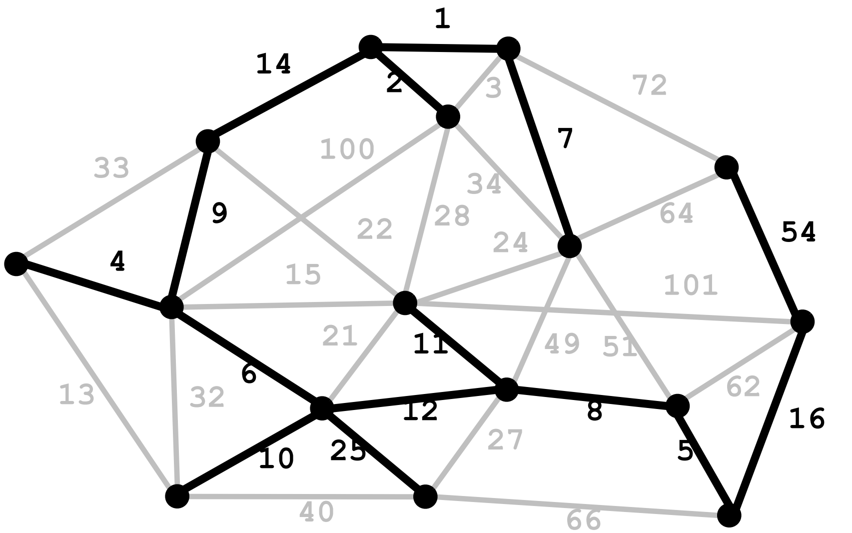
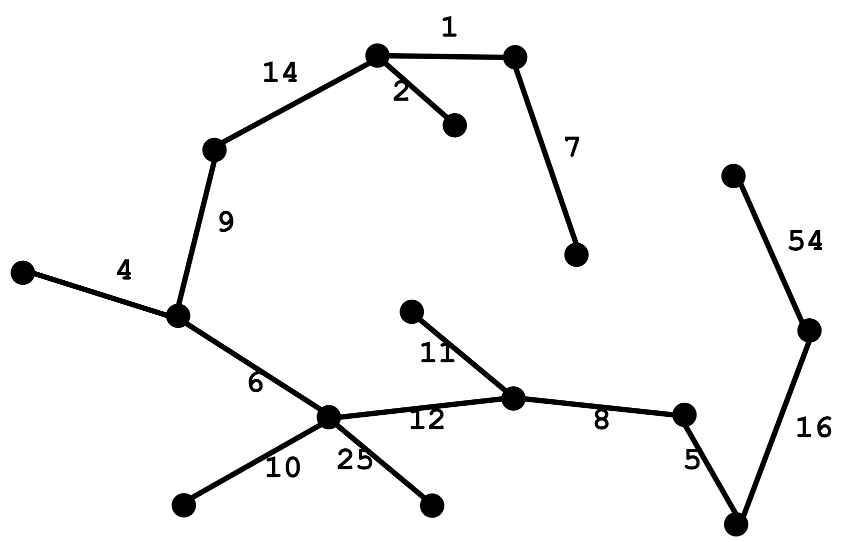


3 Greedy Algorithms

All 3 greedy algorithms give the same minimum spanning tree (assuming distinct edge weights)



Prim's Algorithm (pseudo-code)

```
prim(s) {
   D[s] = 0;
   for (v != s) D[v] = ∞;
   while (some vertices are unmarked) {
      u = unmarked vertex with smallest D;
      mark u;
      for (each v adj to u) {
         D[v] = min(D[v], w(u,v));
      }
   }
}
```

- $O(n^2)$ for adj matrix
  - While-loop is executed n times
  - For-loop takes O(n) time
- O(m + n log n) for adj list
  - Use a PQ
  - Regular PQ produces time O(n + m log m)
  - Can improve to O(m + n log n) using a fancier heap

Greedy Algorithms

- These are examples of Greedy Algorithms
- The Greedy Strategy is an algorithm design technique
  - Like Divide & Conquer
- Greedy algorithms are used to solve optimization problems
  - The goal is to find the *best* solution
- Works when the problem has the greedy-choice property
  - A global optimum can be reached by making locally optimum choices
- Example: the Change Making Problem: Given an amount of money, find the smallest number of coins to make that amount
- Solution: Use a Greedy Algorithm
  - Give as many large coins as you can
- This greedy strategy produces the optimum number of coins for the US coin system
- Different money system $\Rightarrow$ greedy strategy may fail
  - Example: old UK system

Similar Code Structures

```
while (some vertices unmarked) {
   u = best of unmarked vertices;
   mark u;
   for (each v adj to u) {
      update v;
   }
}
```

- BFS
  - best: next in queue
  - update: D[w] = D[v]+1
- Dijkstra
  - best: next in PQ
  - update: D[w] =
    min D[w], D[v]+c(v,w)
- Prim
  - best: next in PQ
  - update: D[w] =
    min D[w], c(v,w)

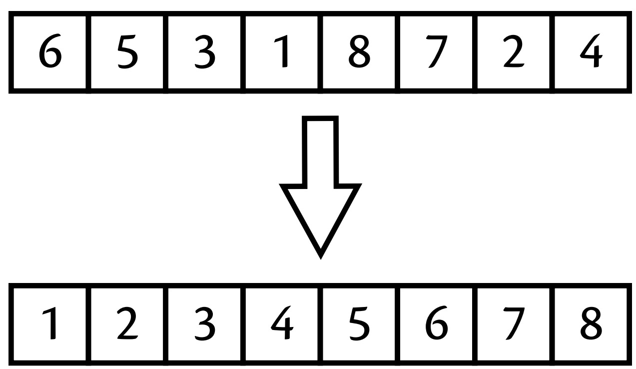
Selection algorithms

6	5	3	1	8	7	2	4

- Find largest?
- Find smallest?
- Find k^{th} smallest?

Quickselect

- Find the k^{th} smallest element of an array



O(n log n)

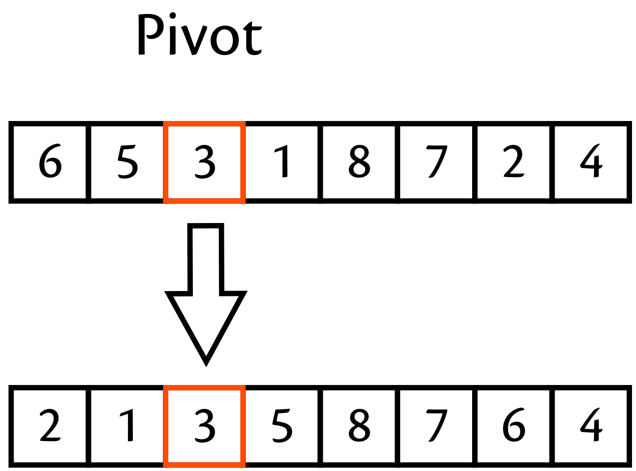
Can we do better?

Pick (N-k)th element

Quickselect

```
function select(list, left, right, k)
     if left = right // return if the list has one element
         return list[left]
     // select pivotIndex between left and right
     pivotNewIndex := partition(list, left, right,
 pivotIndex)
     pivotDist := pivotNewIndex - left + 1
     // The pivot is in its final sorted position,
     // so pivotDist reflects its 1-based position
   // if list were sorted
     if pivotDist = k
         return list[pivotNewIndex]
     else if k < pivotDist
         return select(list, left, pivotNewIndex - 1, k)
     else
         return select(list, pivotNewIndex + 1, right,
                      k - pivotDist)
```

Quickselect



3 is in it's sorted place,
so if k=3 we are done

otherwise, only need 1
recursive call.

Quickselect running time

- O(n) on average, but worst case $O(n^2)$
- Worst case happens with bad pivots
  - Min or max of the elements means we select all elements as pivots

- IDEA: ensure O(n) performance by consistently choosing good pivots

Selection in O(n)

- Best pivot is *median* of elements since it will partition the elements equally
- Median-of-medians algorithm
  - –Divide list into (n/5) groups of five
  - –Find median of each group (n/5 medians)
  - –Then find median these medians
  - –Choose this value as the pivot.

Pivot

- Pivot is less than half of the medians
  - n/10 elements
- Each median is less than 2 elements from its group of 5
- So, the pivot is less than 3(n/10) elements
- Similarly, pivot is greater 3(n/10) elements
- Somewhere between 30/70 and a 70/30 split.
  - Ensures O(n)!