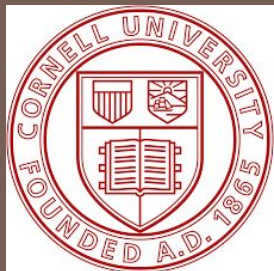
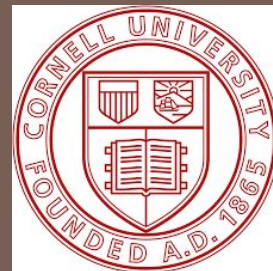


Object-oriented programming and data-structures



CS/ENGRD 2110
SUMMER 2018



Lecture 13: Shortest Path

<http://courses.cs.cornell.edu/cs2110/2018su>

Graph Algorithms

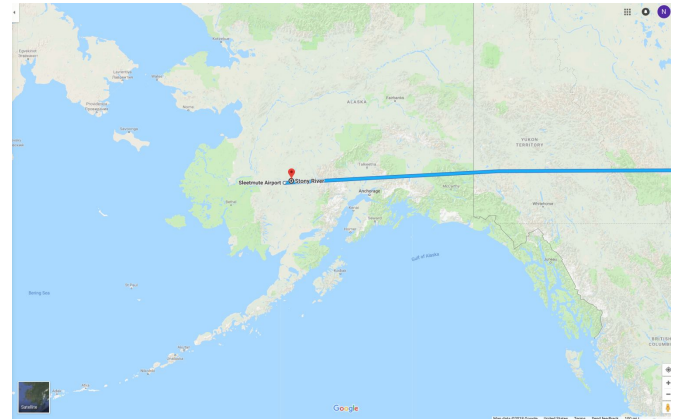
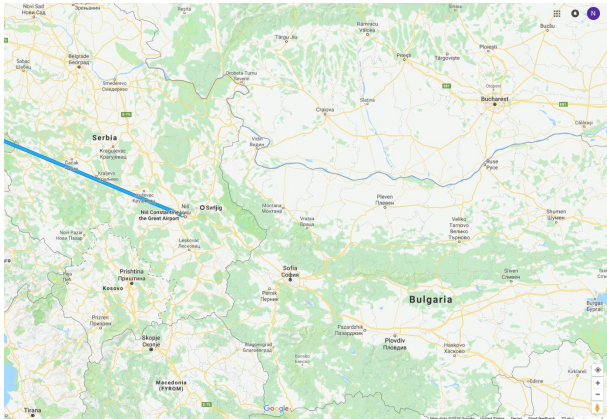
- Search
 - Depth-first search
 - Breadth-first search
 - Shortest paths
 - Dijkstra's algorithm
- Spanning trees
 - Prim's algorithm
 - Kruskal's algorithm

Shortest Path Problem

- How do I efficiently find the shortest path from s to v in a graph?

Shortest Path Problem

- How do I efficiently find the shortest path from s to v in a graph?
- What is the shortest path to fly from Svrljig (Serbia, Population: 7533) to Stony River (Alaska, USA, Population: 52)



Shortest Path Problem

- Shortest path between Svrlijg to Stony River requires **8 hops**

The screenshot shows a flight search interface for a round trip from Svrlijg, Serbia to Stony River, SRV. The search parameters are set for a round trip, 1 passenger, and Economy class. The departure date is Thursday, August 9, and the return date is Wednesday, August 15.

Flight insights

- DATES:** Cheaper flights from \$3,922 available on other dates. [SEE MORE](#)
- PRICE GRAPH:** Explore price trends for 6-day trips to Stony River. [SEE MORE](#)
- AIRPORTS:** Compare prices for airports near Stony River. [SEE MORE](#)
- TIPS:** Fly in Business for \$9,340. [SEE MORE](#)

Best departing flights

Total price includes taxes + fees for 1 adult. [Additional bag fees](#) and other fees may apply. Sort by: \uparrow

Flight	Time	Stops	Price
Air Serbia, British Airways, Alaska, Ravn Alaska	10:25 AM - 1:40 PM ^{*1}	37h 15m BEG-SRV	8 stops ▲ LHR, SEA, ANC, ANI, CHU, CKE \$4,024 round trip
Lufthansa, Alaska, Ravn Alaska	6:30 AM - 1:40 PM ^{*1}	41h 10m BEG-SRV	8 stops ▲ FRA, SEA, ANC, ANI, CHU, CKE \$5,216 round trip

Track prices

Monitor the lowest price for this trip, and receive price alerts and travel tips by email.

Other departing flights

Flight	Time	Stops	Price
Air Serbia, Delta, Ravn Alaska	6:40 AM - 1:40 PM ^{*1}	41h 0m BEG-SRV	8 stops ▲ AMS, MSP, ANC, ANI, CHU, CK \$13,064 round trip
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Shortest Path Problem

- Shortest path between Svrlijg to Stony River requires **8 hops**
- Google Flights computed this is a few milliseconds. Billions of possible paths!
- Have we seen an algorithm that can compute the shortest path?

The screenshot shows a Google Flights search interface. At the top, it displays 'Round trip', '1 passenger', and 'Economy'. The search parameters are 'Svrlijg, Serbia' to 'Stony River, SRV' for dates 'Thu, Aug 9' to 'Wed, Aug 15'. Below the search bar, there are tabs for 'Choose departure to Stony River', 'Return to Svrlijg, Serbia', and 'Trip summary'. The main content area is titled 'Flight insights' and includes four panels: 'DATES' (Cheaper flights from \$3,922), 'PRICE GRAPH' (Explore price trends for 6-day trips), 'AIRPORTS' (Compare prices for airports near Stony River), and 'TIPS' (Fly in Business for \$9,340). Below this is the 'Best departing flights' section, which lists two flight options. The first option is a round trip from LHR to SEA, ANG, ANI, CHU, CKE via Air Serbia, British Airways, Alaska, and Ravn Alaska, with a duration of 37h 15m and a price of \$4,024. The second option is a round trip from FRA to SEA, ANG, ANI, CHU, CKE via Lufthansa, Alaska, Ravn Alaska, and United, with a duration of 41h 10m and a price of \$5,216. There is also a 'Track prices' section and an 'Other departing flights' section at the bottom.

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What about BFS

- BFS expands the graph in “layers”
 - First explores all nodes at distance 1 from the source
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What about BFS

- BFS expands the graph in “layers”
 - First explores all nodes at distance 1 from the source
 - Next explores all nodes at distance 2 from the source, etc.
- But BFS only finds the path with the **smallest number of hops**
- Instead, we want to consider **weighted graphs**

Weighted Graphs

- In real graphs, want to assign **weights** to a graph
 - Price
 - Distance
 - Number of miles
- The shortest path is the **path with the lowest weight**, not necessarily the path with the smallest number of **edges**

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The screenshot shows a flight search interface for a round trip from New York City to Paris. The search parameters are set to 1 passenger in Economy class, departing on Thursday, August 9, and returning on Wednesday, August 15. The interface displays flight insights, best departing flights, and a track prices option.

Flight insights

DATES	PRICE GRAPH	AIRPORTS	TIPS
Cheaper flights from \$530 available on other dates SEE MORE	Explore price trends for 6-day trips to Paris SEE MORE	Compare prices for airports near Paris SEE MORE	Fly in Premium Economy for \$1,235 SEE MORE

Best departing flights ⓘ
Total price includes taxes + fees for 1 adult. [Additional bag fees](#) and other fees may apply. Sort by: ⚙️

12:40 AM – 5:25 PM WOW	10h 45m JFK–CDG	1 stop 1h 15m KEF	\$620 round trip
1:40 AM – 2:45 PM XL Airways	7h 5m JFK–CDG	Nonstop	\$686 round trip
11:55 PM – 1:00 PM ⁺¹ XL Airways	7h 5m JFK–CDG	Nonstop	\$763 round trip
4:20 PM – 5:45 AM ⁺¹ Air France · Delta	7h 25m JFK–CDG	Nonstop	\$1,001 round trip
9:55 PM – 11:20 AM ⁺¹ Air France · Delta	7h 25m JFK–CDG	Nonstop	\$1,001 round trip

Track prices
Monitor the lowest price for this trip, and receive price alerts and travel tips by email

Weighted Graphs, formally

- A **weighted** directed graph $G = (V, E, W)$
 - V is a (finite) set
 - E is a set of *ordered* pairs (u, v) where $u, v \in V$
 - W is *weight function that assigns edges to real-valued **weights***

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- A **weighted** directed graph $G = (V, E, W)$
 - V is a (finite) set
 - E is a set of *ordered* pairs (u, v) where $u, v \in V$
 - W is *weight function that assigns edges to real-valued **weights***
- Recall that a path is a sequence of edges $p = (v_0, v_1, v_2, \dots, v_k)$
 - The weight $w(p)$ of a path $p = (v_0, v_1, v_2, \dots, v_k)$ is the sum of the weights of its constituent edges

- $$w(p) = \sum_{i=1}^k w(v_{i-1}, v_i)$$

Scoping the Problem

- Single Destination Shortest Paths Problem
 - Find a shortest path between two vertices **u and v**

Scoping the Problem

- Single Destination Shortest Paths Problem
 - Find a shortest path between two vertices **u and v**
- All-pairs shortest path problem
 - Find a shortest path from u to v for every pair of vertices u and v
 - Can run case-above for all vertices u and v
 - But exists a more efficient algorithm (Floyd-Warshall Algorithm)
 - **We do not look at this in this class!**

Single-Source Shortest Path (SSSP)

- Two algorithms:
 - Dijkstra's Algorithm
 - Bellman Ford Algorithm
- Dijkstra's algorithm has complexity $O(V+E)$
- Bellman-Ford's algorithm has complexity $O(VE)$
- Dijkstra works only for **positive edges**. Bellman-Ford works for both **positive and negative edges**.
- In this class we will only look at Dijkstra's algorithm!

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Shortest Path - Definition

- We define the **shortest path weight** $\delta(u,v)$ from u to v by:

$$w(p) = \begin{cases} \min(w(p) : u \rightsquigarrow v) & \text{If there is a path from } u \text{ to } v \\ \infty & \text{Otherwise} \end{cases}$$

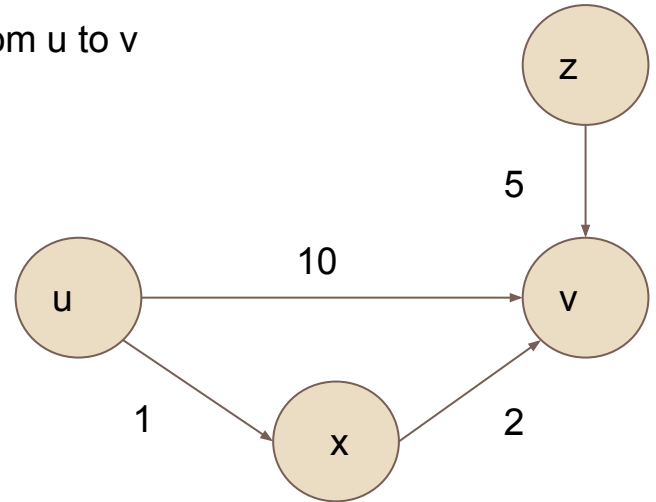
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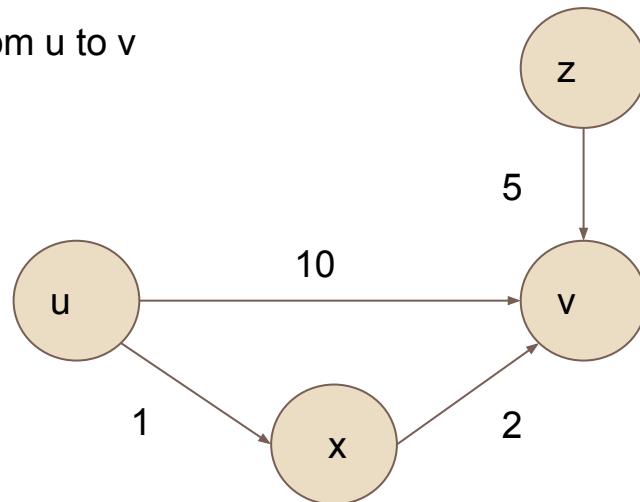
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$$\delta(u,v) = ?$$

$$\delta(z,v) = ?$$

$$\delta(z,u) = ?$$



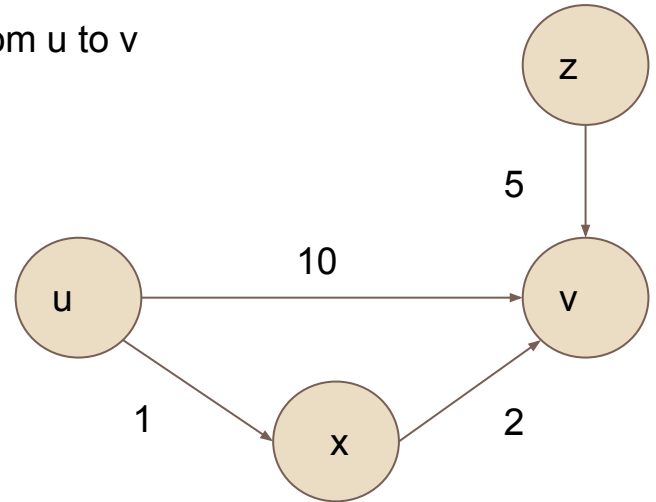
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$$\begin{aligned} \delta(u,v) &= 3 \\ \delta(z,v) &= 5 \\ \delta(z,u) &= \infty \end{aligned}$$

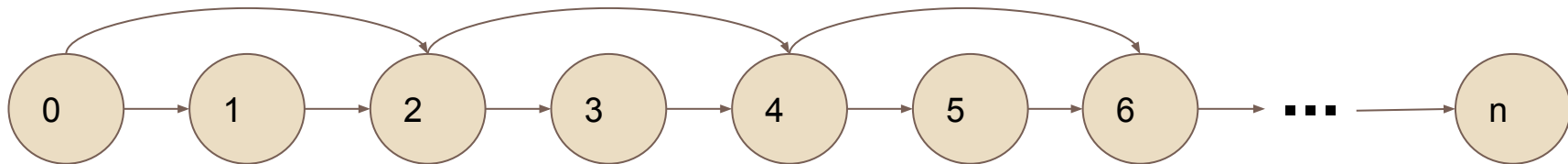


What about brute-force?

- What if we simply enumerated all paths between u and v , and picked the one with the smallest weight?
- How many paths between two nodes can there be in the worst-case?

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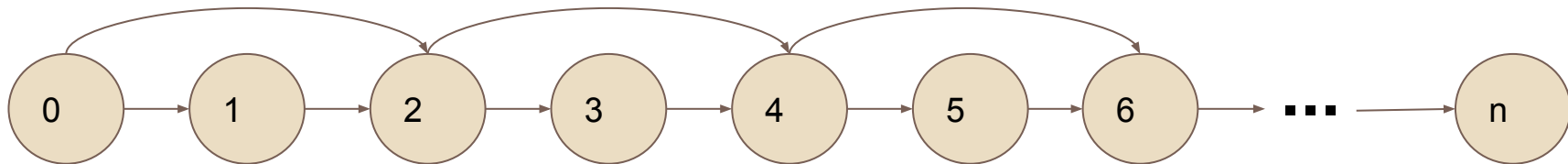
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Paths from 0 to 1?

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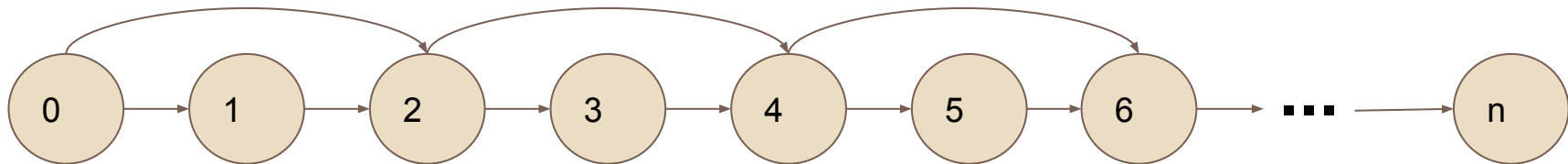
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Paths from 0 to 1? 1

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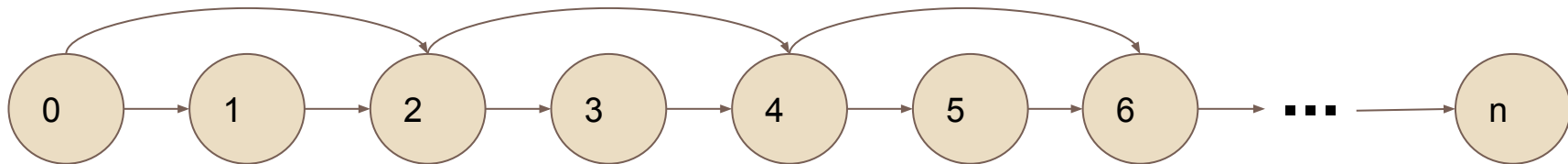


Paths from 0 to 1? 1

Paths from 0 to 2?

What about brute-force?

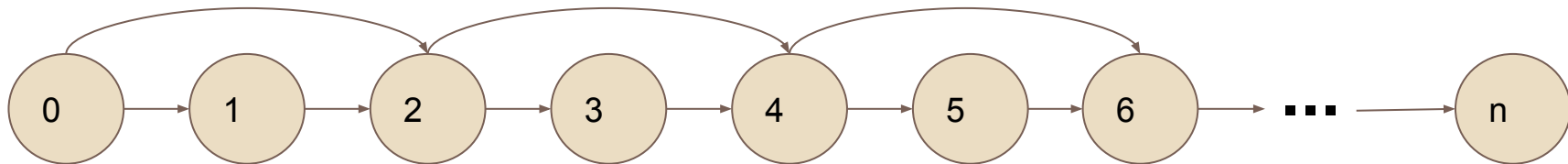
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Paths from 0 to 1? 1
Paths from 0 to 2? 2

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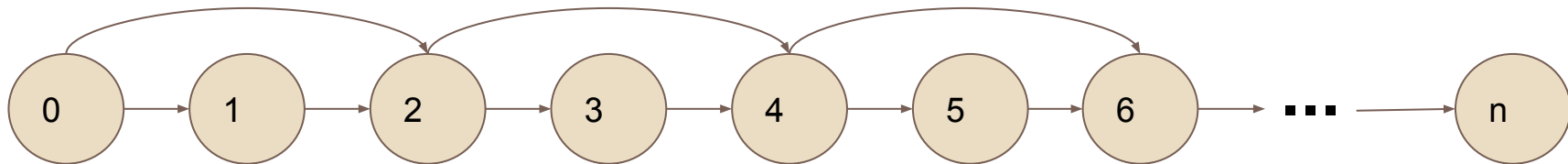
Paths from 0 to 1? 1

Paths from 0 to 2? 2

Paths from 0 to 4? 4

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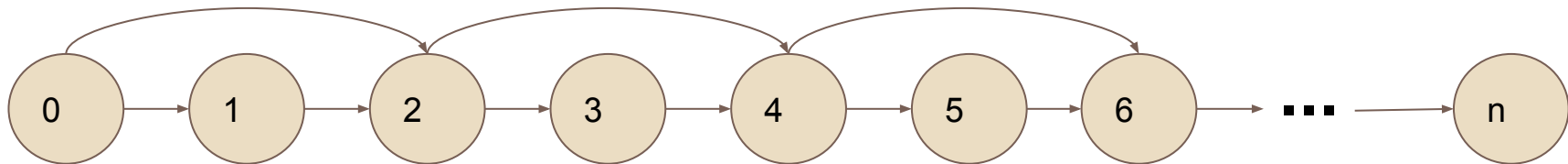
Paths from 0 to 2? 2

Paths from 0 to 4? 4

Paths from 0 to 6? 8

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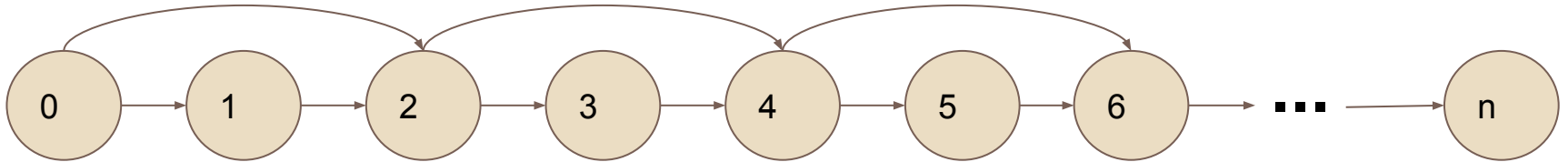
Paths from 0 to 4? 4

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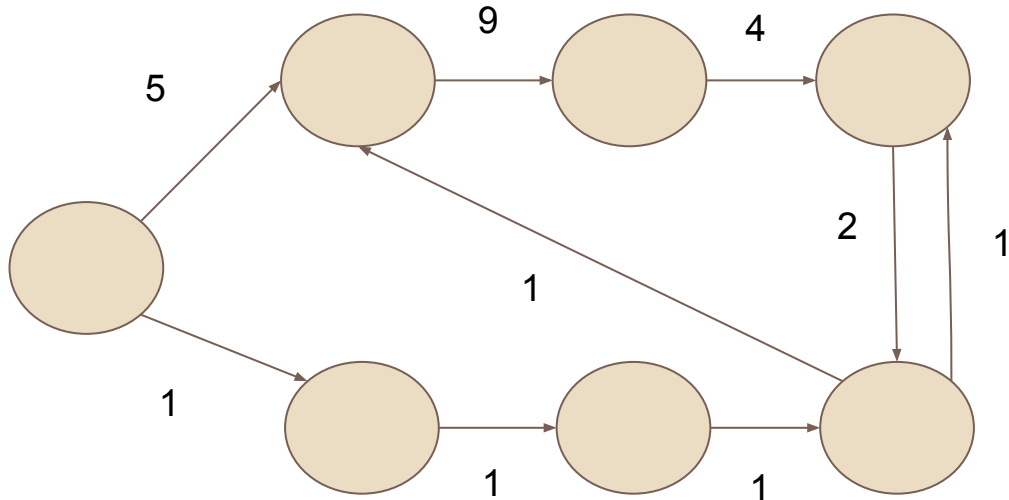


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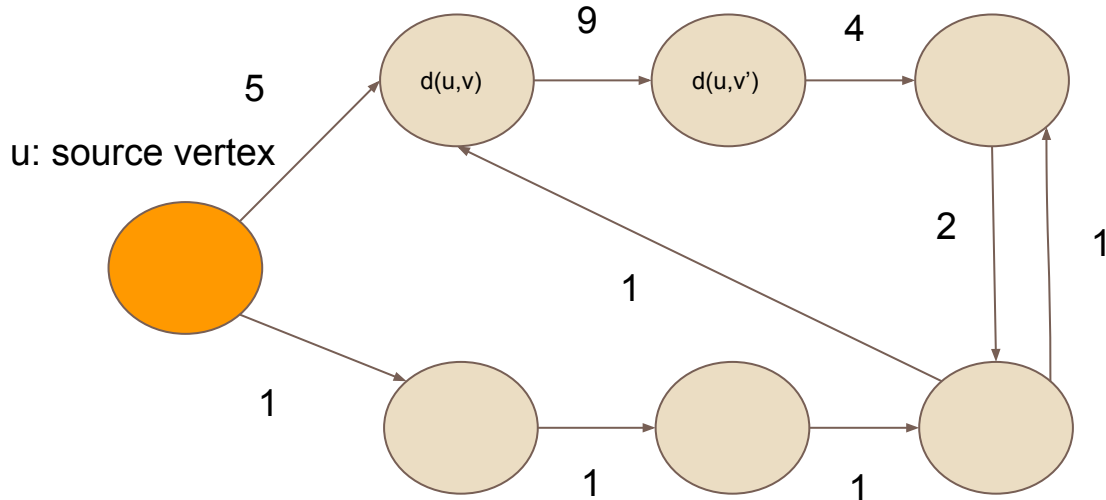
Order $2^{(n/2)}$

Exponentially many paths

Terminology

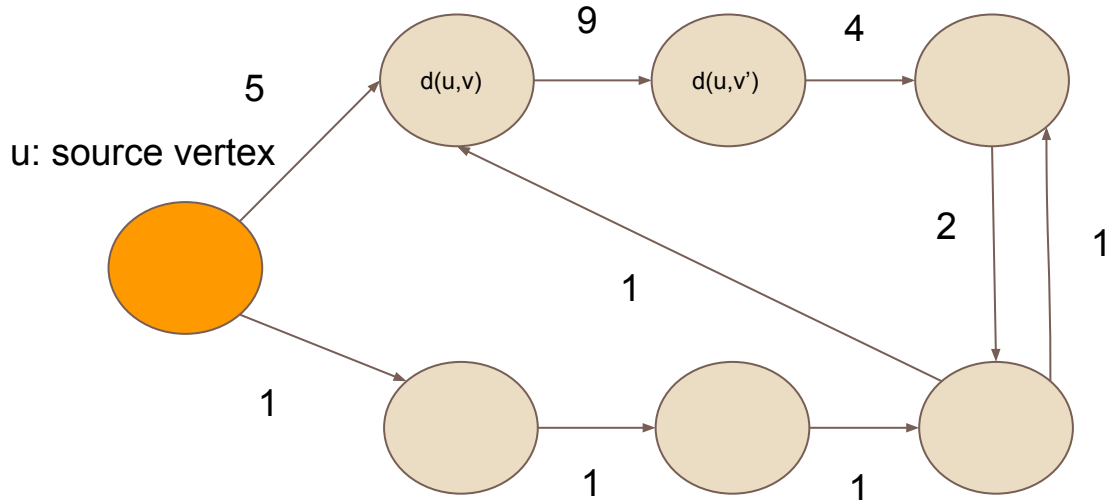


Terminology - Current Weight



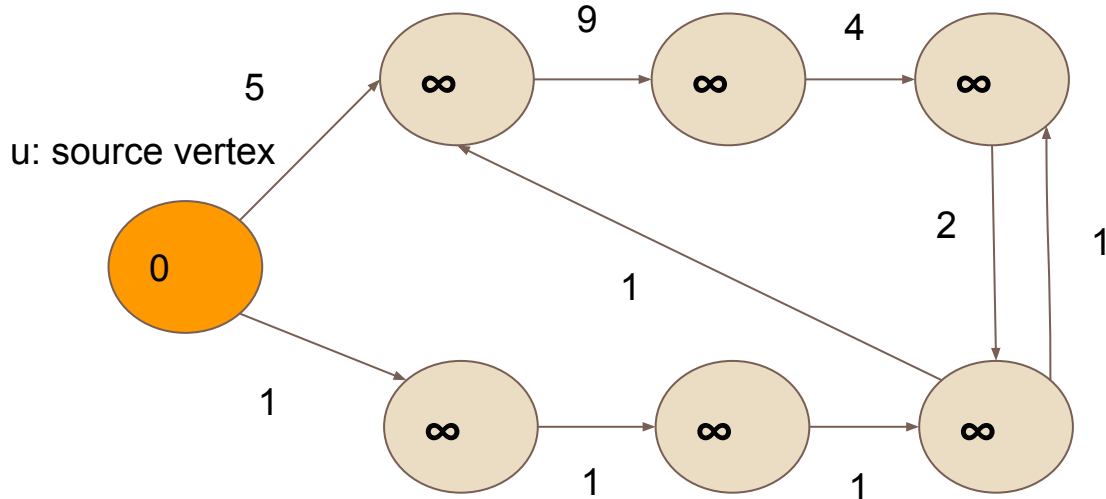
Write $d(u,v)$ to be the **current weight** of node v : it represents the current best estimate of the shortest path from u to v

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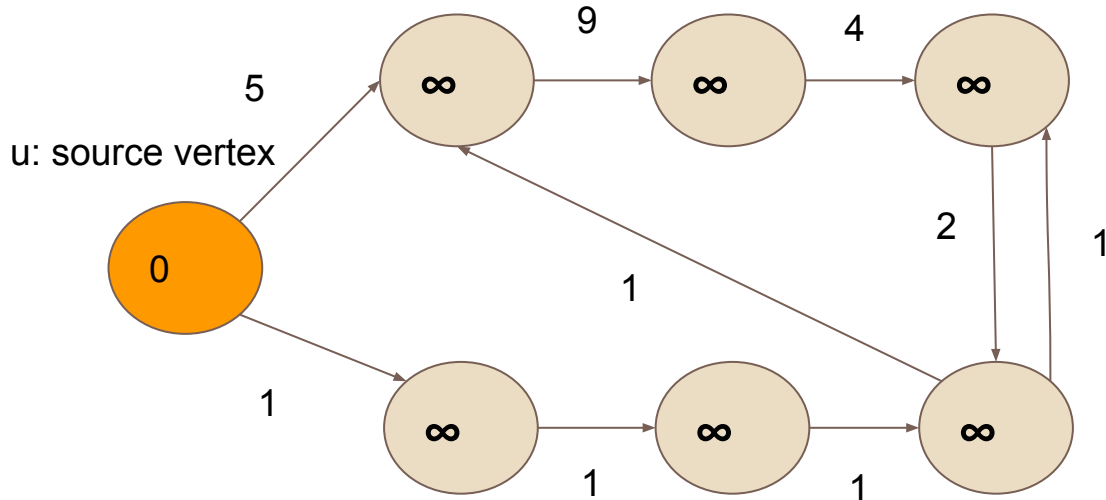
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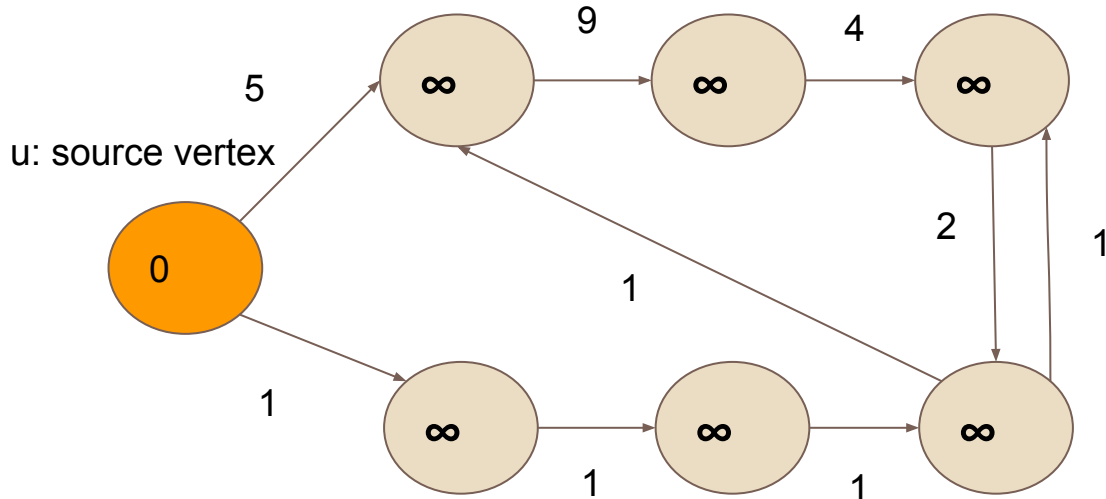


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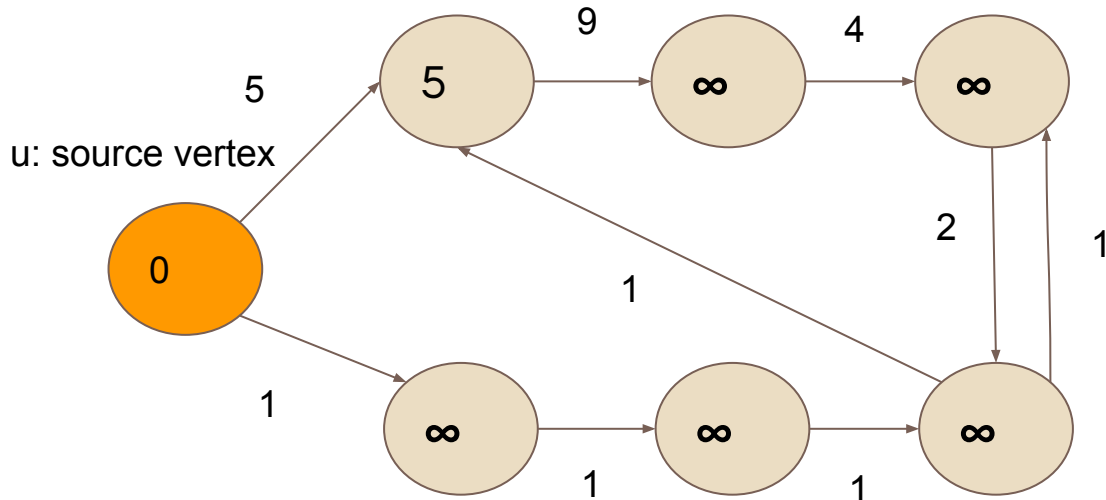
Goal: reduce $d(u)$ until sure that $d(u) = \delta(u,v)$

Terminology - Path Relaxation



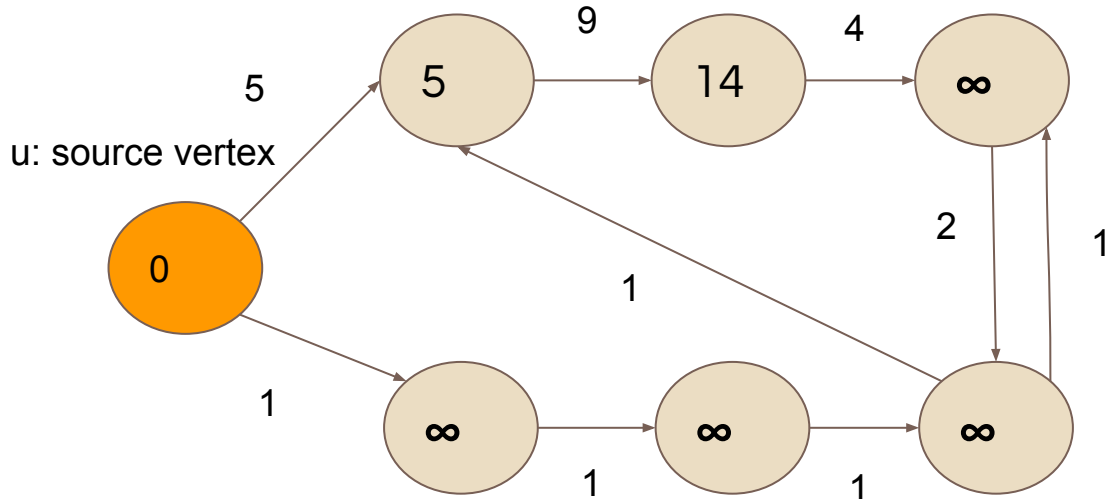
As discover new **paths**, will **update estimates** of what is currently the shortest path

Terminology - Path Relaxation



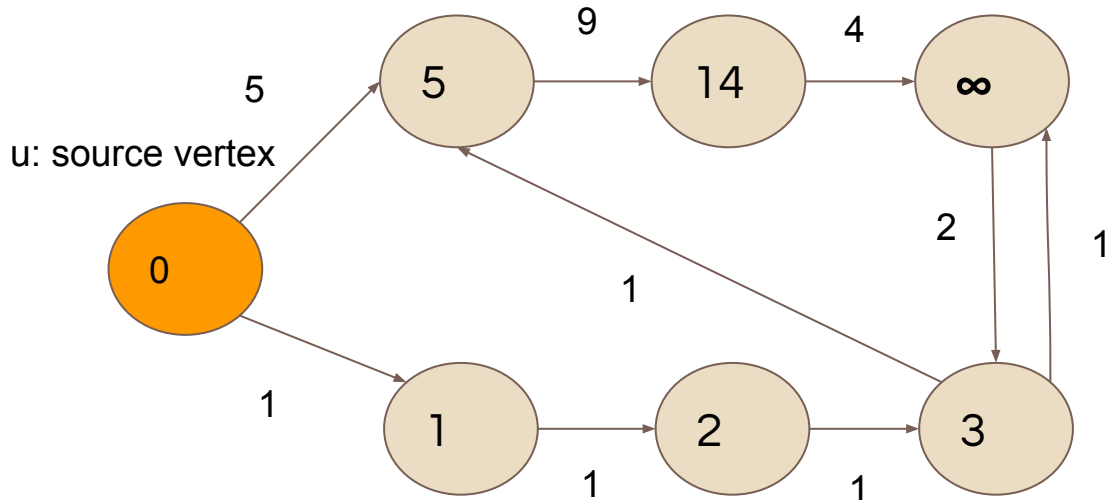
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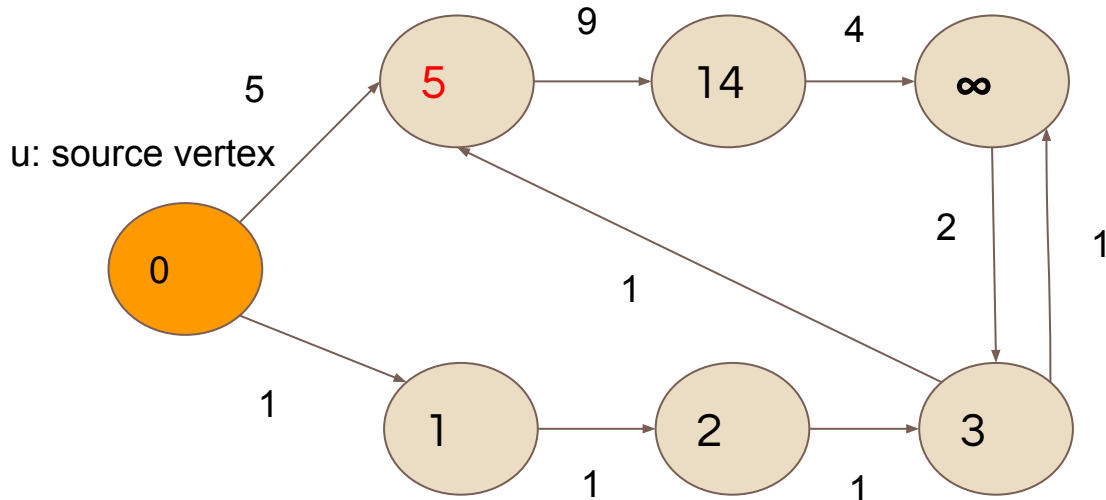
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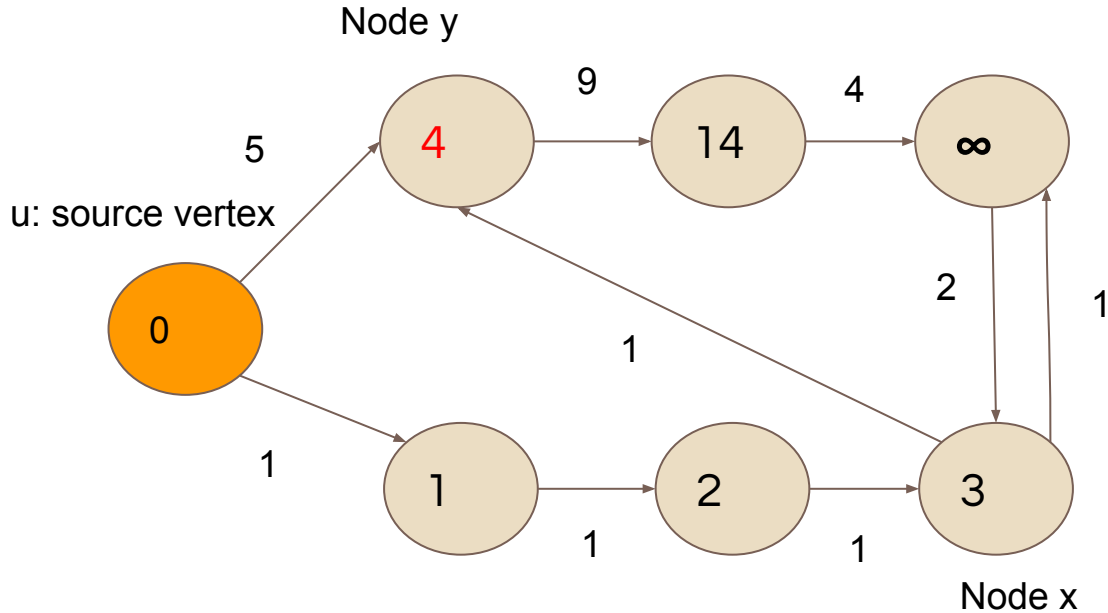
Terminology - Path Relaxation



Path relaxation:

Given a new edge (u,v) :
If $d[u] + w(u,v) < d[v]$, then
we have discovered a
better way to get from s to
 v , so update $d[v] = d[u] +$
 $w(u,v)$

Terminology - Predecessor



Keep track of the **predecessor of a node**: the node u that precedes v in the current estimate of the shortest path

$$\pi[y] = x$$

Initially $\pi[y] = \text{null}$

During path relaxation, if $d[u] + w(u,v) < d[v]$, then update $\pi[v] = u$

General Structure of SSSP

- **Initialisation**
 - For u in V : $d[v] = ? \pi[u] = ?$
 - $d[s] = ?$

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- Repeat until **[When?]**
 - Select some edge (u,v) **[How?]**
 - Relax edge (u,v) :
 - if $d[v] > d[u] + w[u,v]$
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Checking whether edges can be relaxed is $O(E)$. Expensive!

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How many iterations will this do in the worst case?

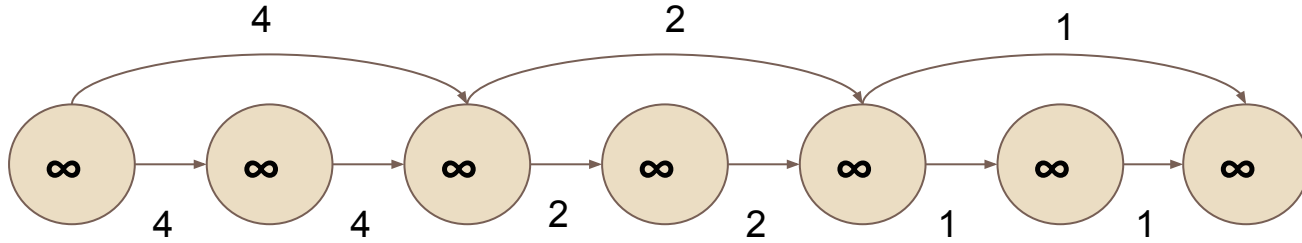
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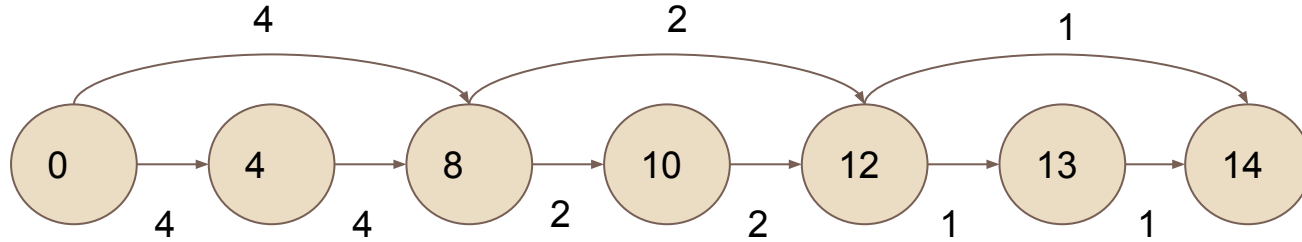
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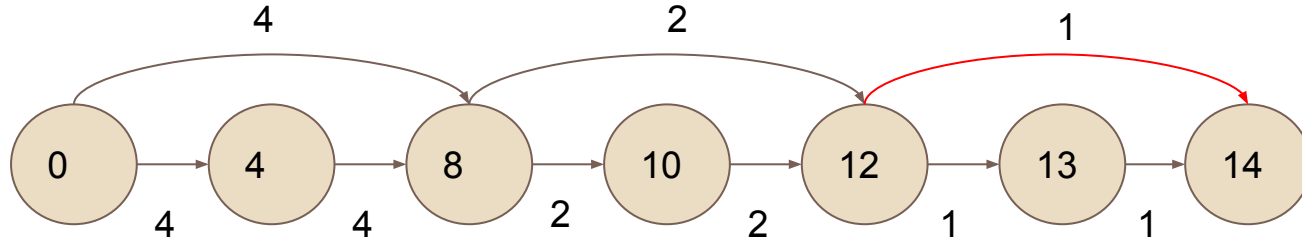
Worst-Case Iterations



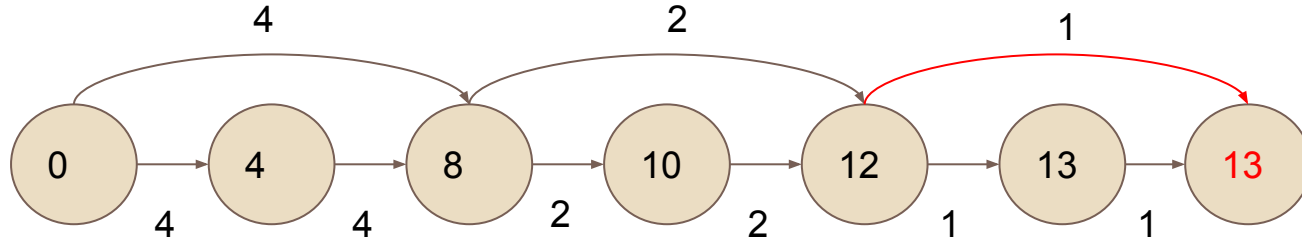
Worst-Case Iterations



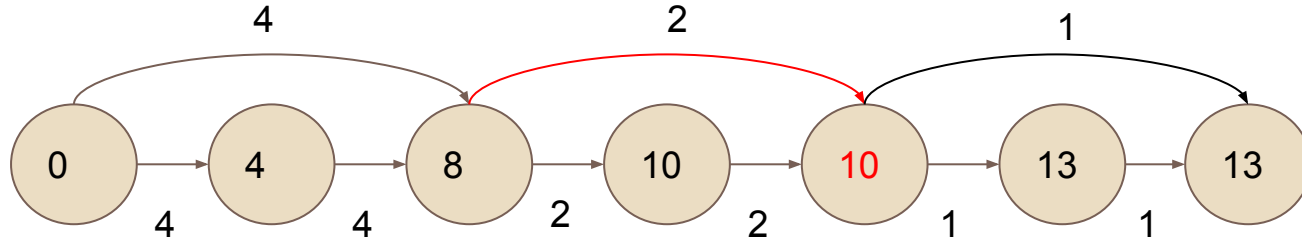
Worst-Case Iterations



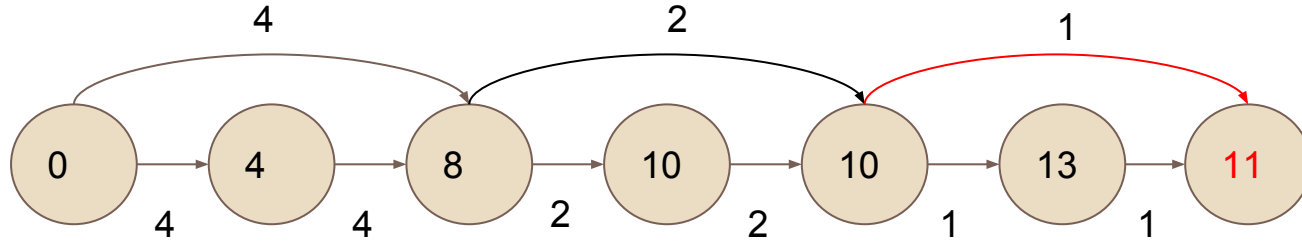
Worst-Case Iterations



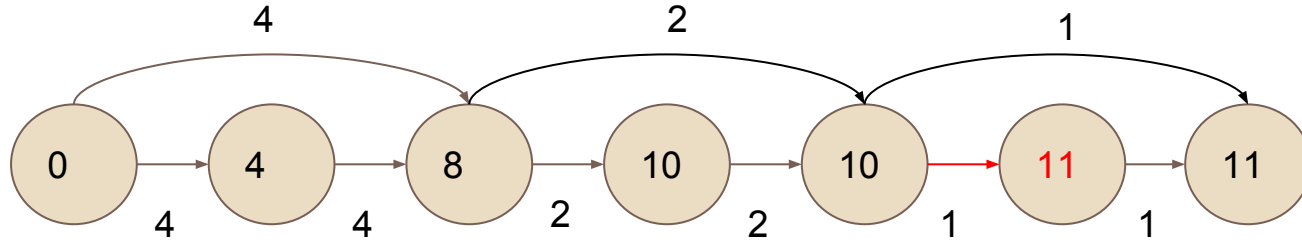
Worst-Case Iterations



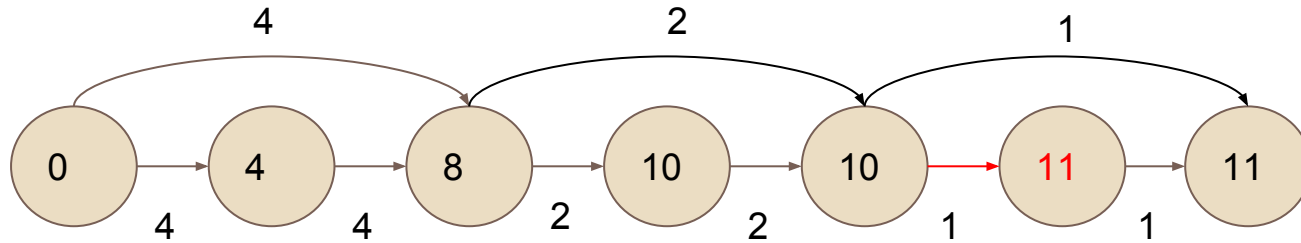
Worst-Case Iterations



Worst-Case Iterations



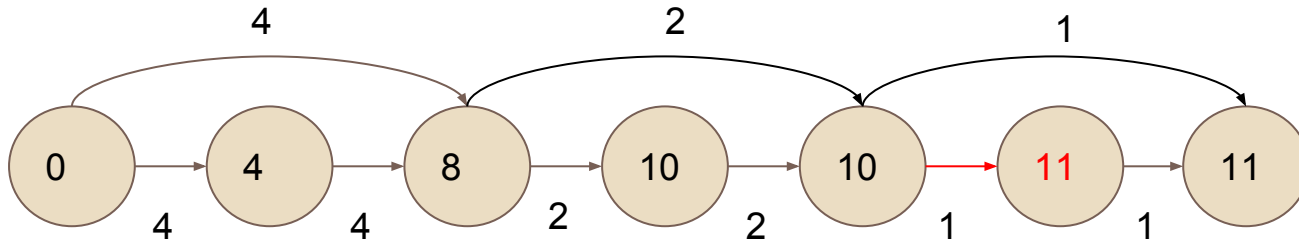
Worst-Case Iterations



Keep going decrementing from 13 (initial value), until shortest path value of 7

How many iterations does this take?

Worst-Case Iterations



Keep going decrementing from 13 (initial value), until shortest path value of 7

How many iterations does this take? $2^{n/2}$...

We have an exponential algorithm! (Again!)

Need to find some way to “intelligently” select the edges.

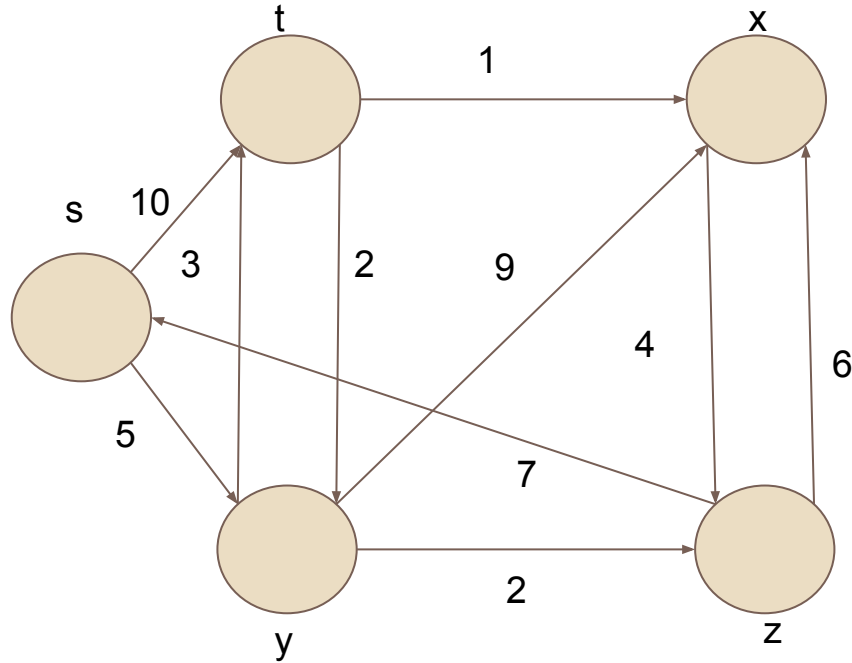
Dijkstra's algorithm

- We need a way to **bound** the number of times that we relax edges
- Dijkstra's algorithm does this by **greedily** selecting the vertex v with the smallest $d(u,v)$ and **relaxing** its neighbouring edges.
- We'll see how this is sufficient to guarantee that $d(u,v) = \delta(u,v)$ once all vertices have been processed
- It only requires 1 pass on all the vertices (V) and all the edges (E)!
- The algorithm itself is surprisingly simple. The proof is harder.

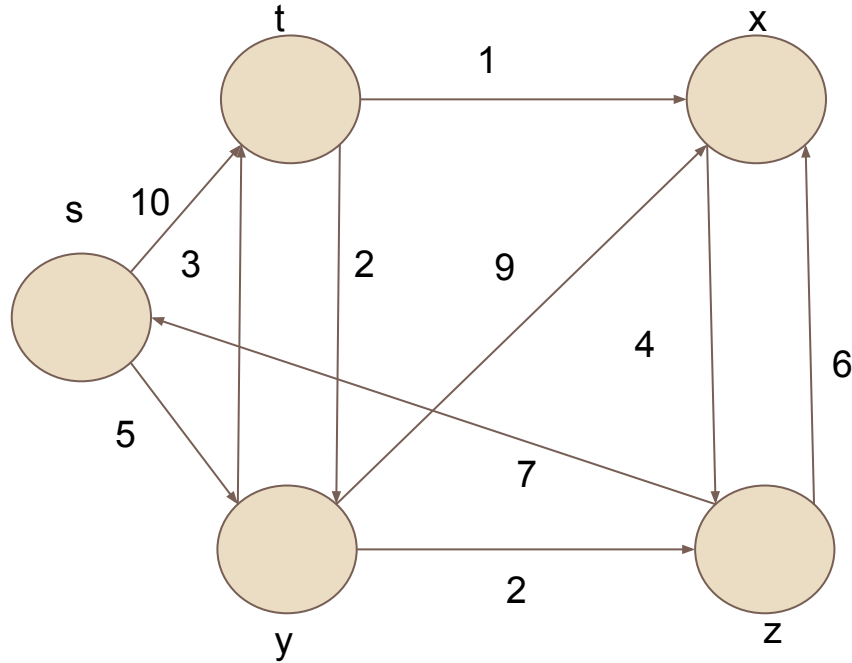
Dijkstra's algorithm

- Maintains a set **S** of vertices whose final shortest path weights from source s have already been determined, and a set **Q** of vertices whose shortest path weights are not yet known.
- Algorithm repeatedly selects the vertex v in **Q** with the minimum shortest path estimate.
 - Adds v to **S**.
 - Relaxes all the edges leaving v .
- We'll show in the proof that, at the point where we add v to **S** $d(u,v) = \delta(u,v)$

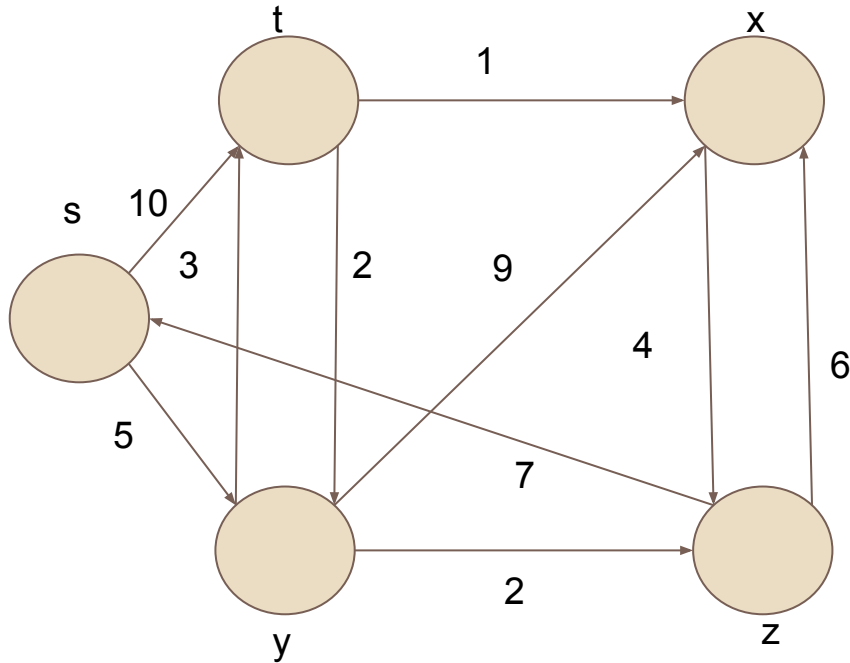
Dijkstra's algorithm



Dijkstra's algorithm



Dijkstra's algorithm



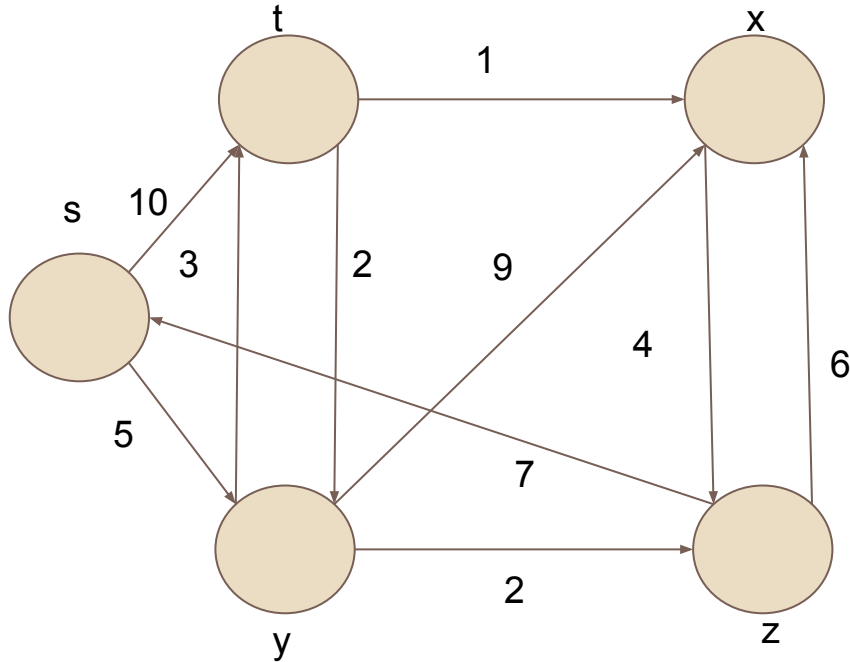
Initialisation

$d[s,s] = ?$

$d[s,t] = ?$

$d[s,x] = ?$

Dijkstra's algorithm



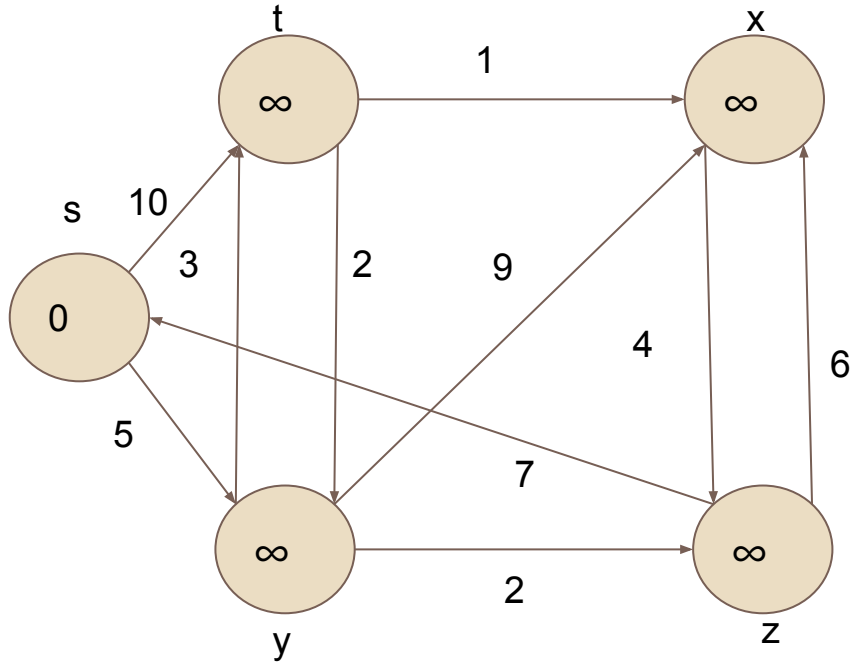
Initialisation

$$d[s,s] = 0$$

$$d[s,t] = \infty$$

$$d[s,x] = \infty$$

Dijkstra's algorithm



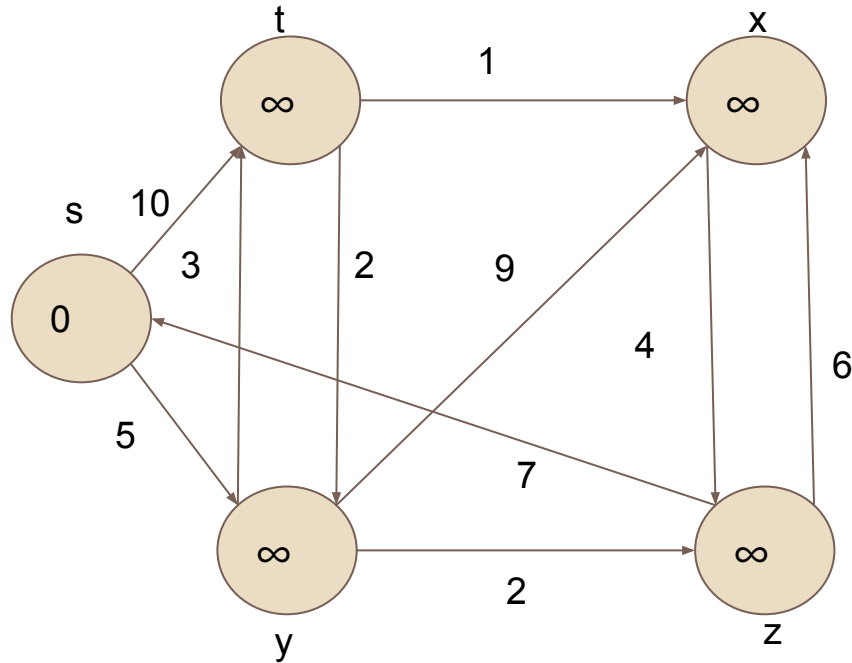
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Dijkstra's algorithm



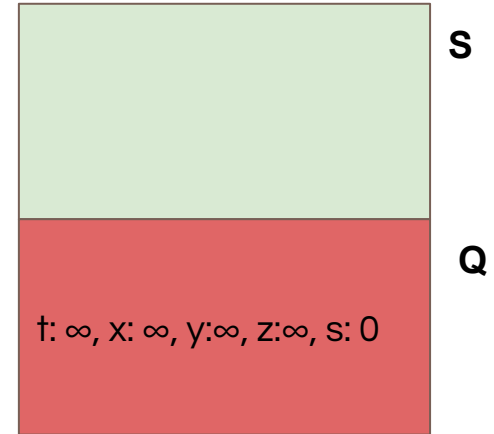
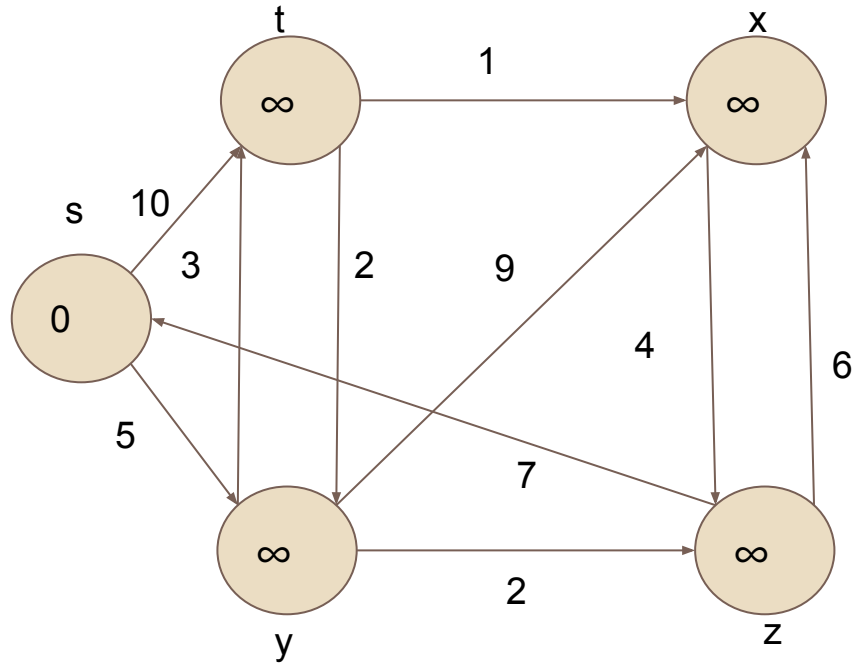
Initialisation

$$d[s,s] = 0 \quad \Pi[s] = \text{null}$$

$$d[s,t] = \infty \quad \Pi[t] = \text{null}$$

$$d[s,x] = \infty \quad \Pi[x] = \text{null}$$

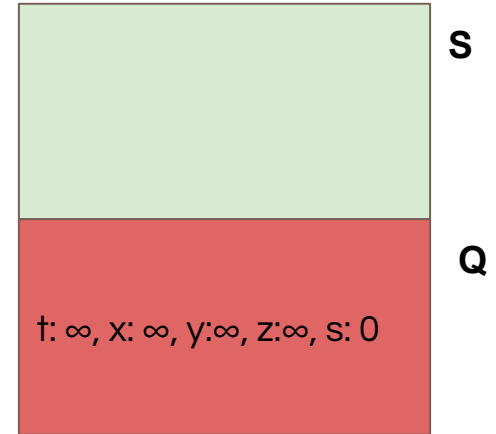
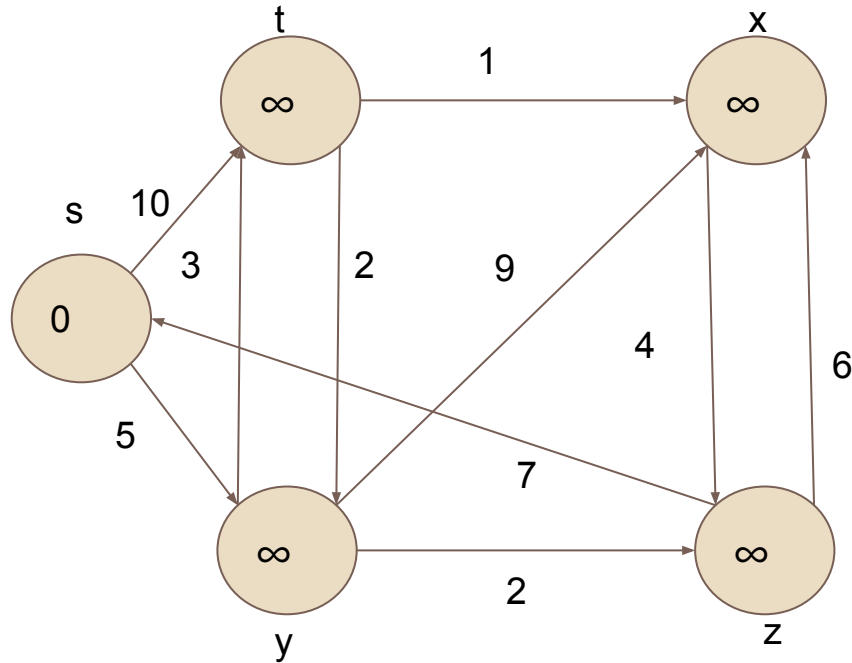
Dijkstra's algorithm



Initialisation

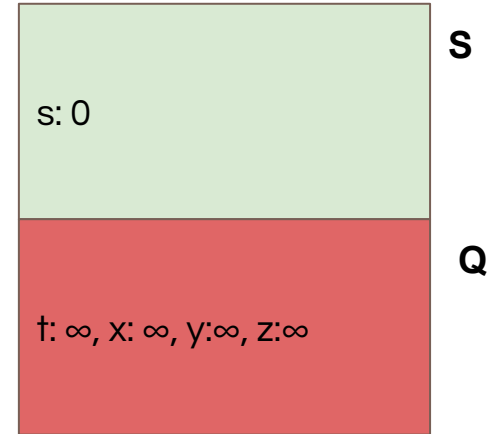
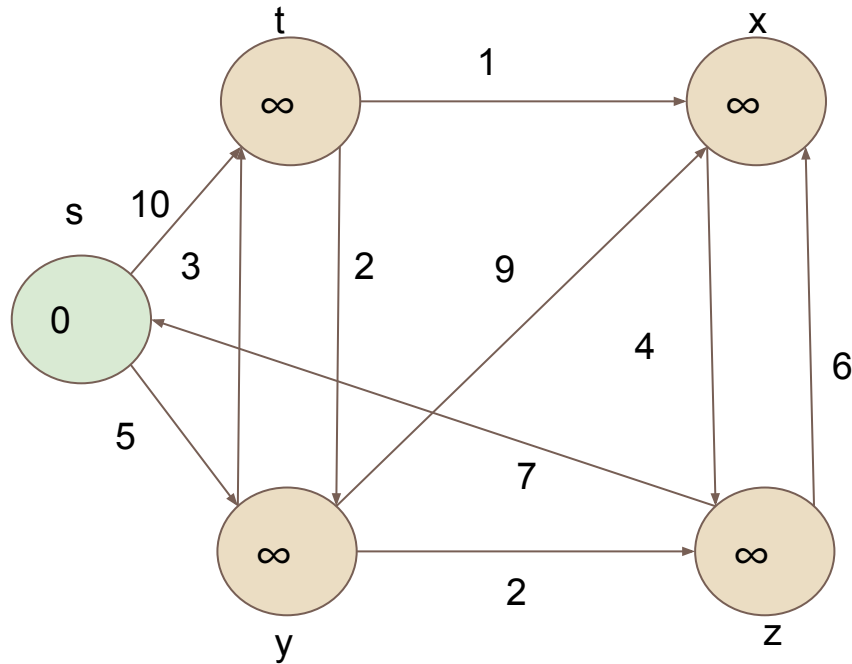
Place all node V in Q.

Dijkstra's algorithm



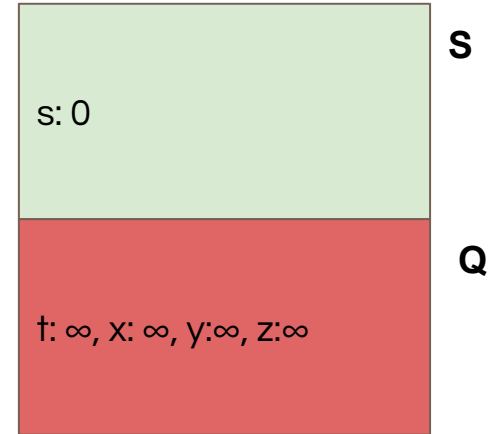
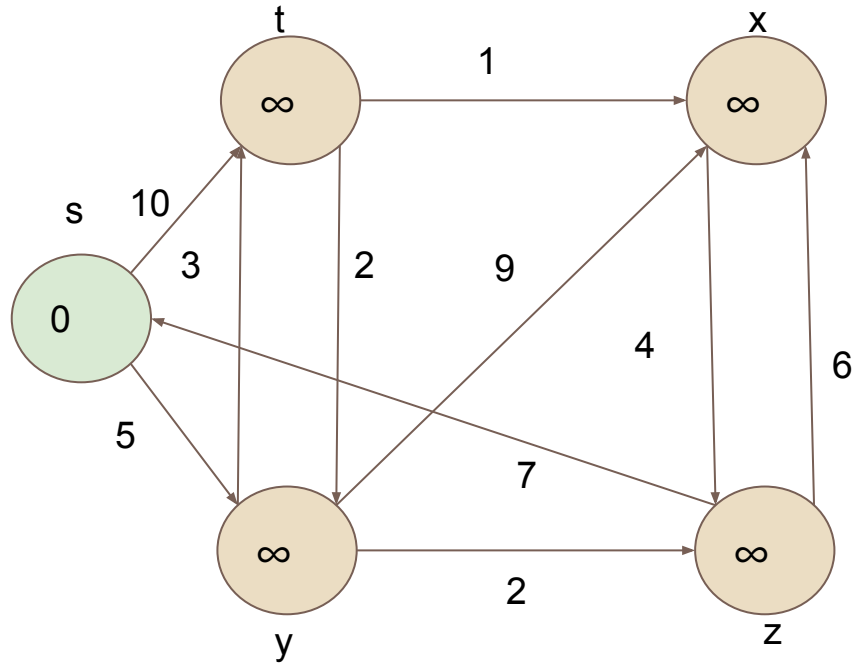
Pick node with smallest $d[s,v]$ and place it in S

Dijkstra's algorithm



Pick node with smallest $d[s,v]$ and place it in S

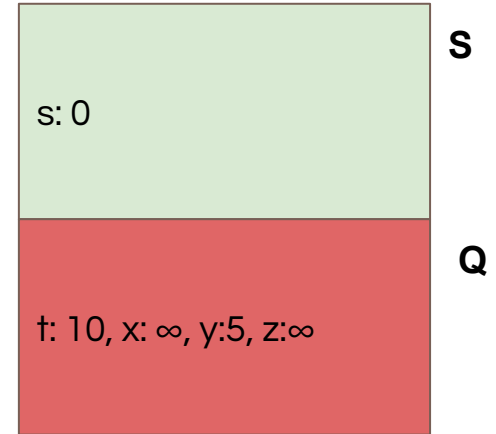
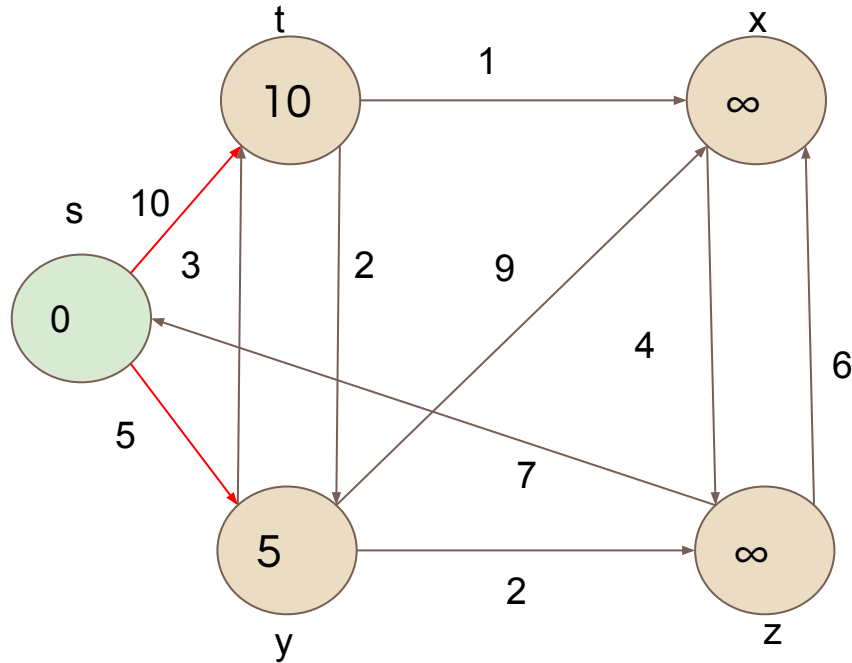
Dijkstra's algorithm



Pick node with smallest $d[s,v]$ and place it in S

Relax all of its edges

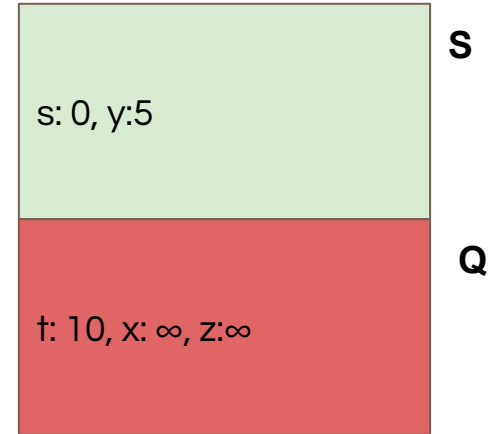
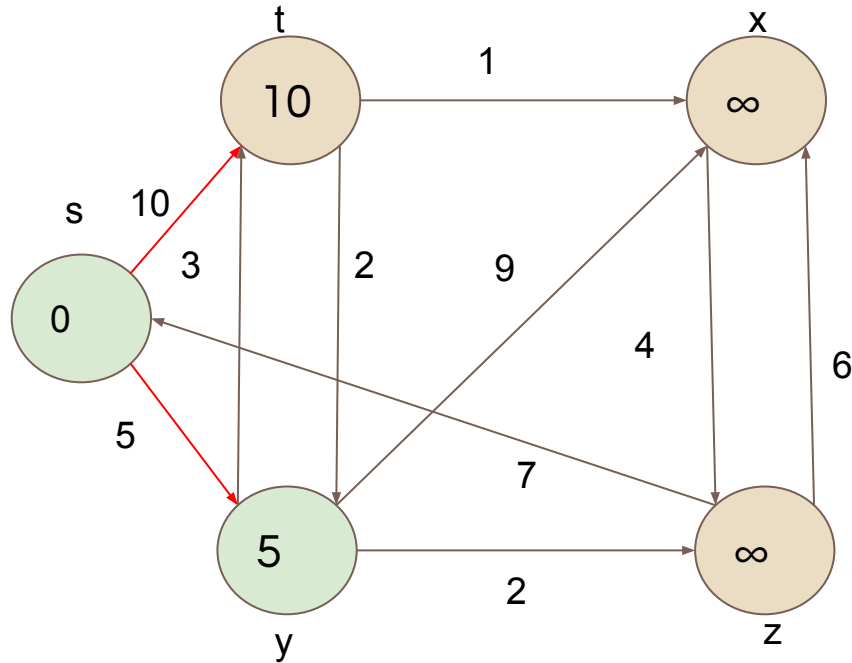
Dijkstra's algorithm



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Relax all of its edges

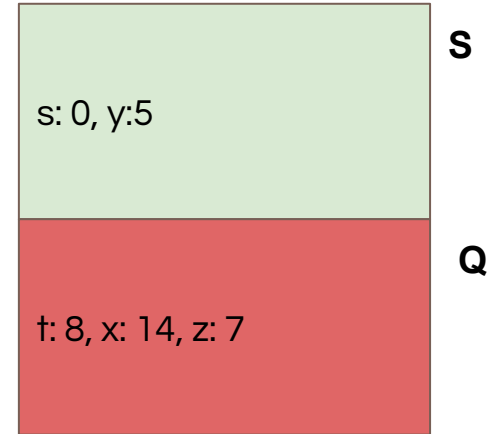
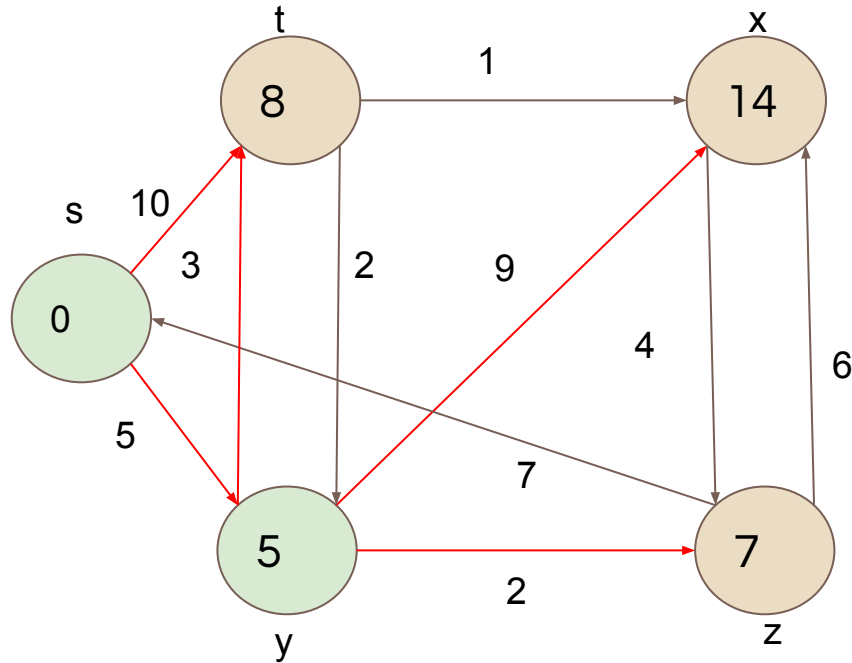
Dijkstra's algorithm



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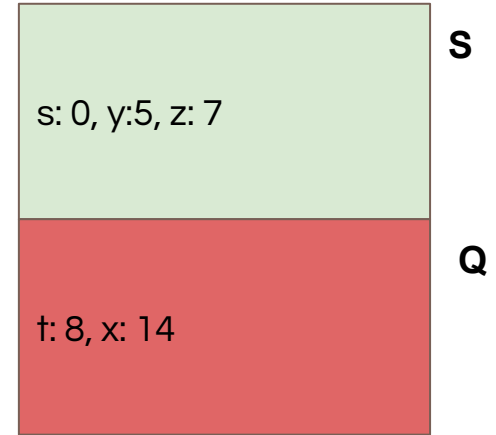
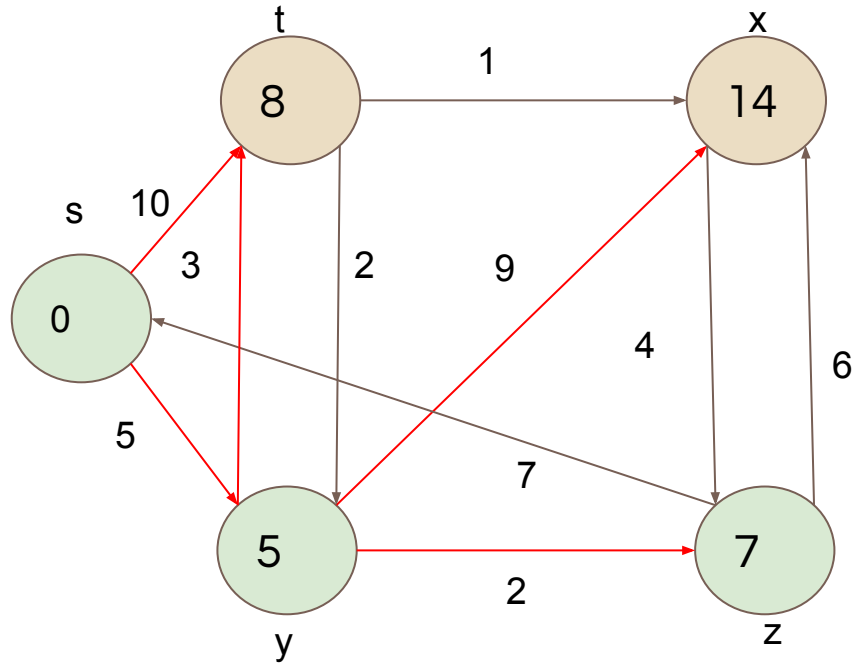
Dijkstra's algorithm



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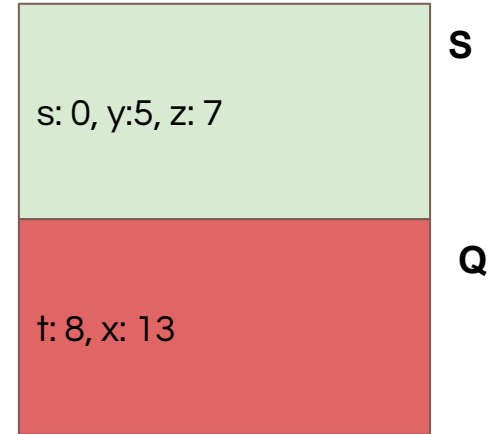
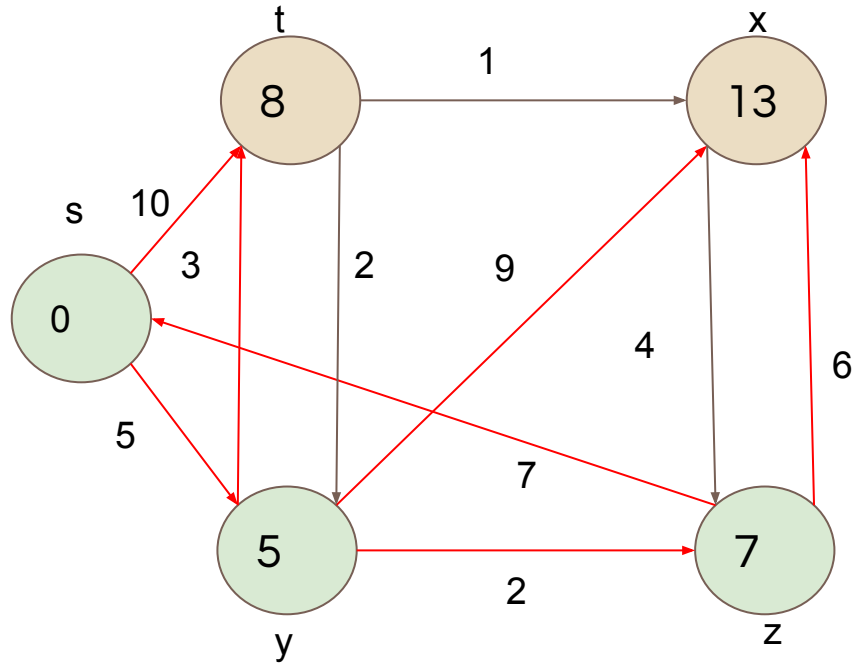
Dijkstra's algorithm



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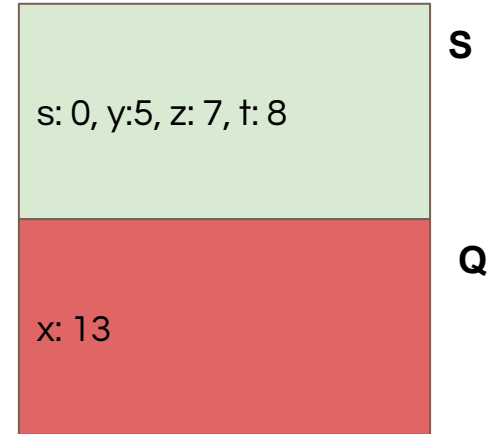
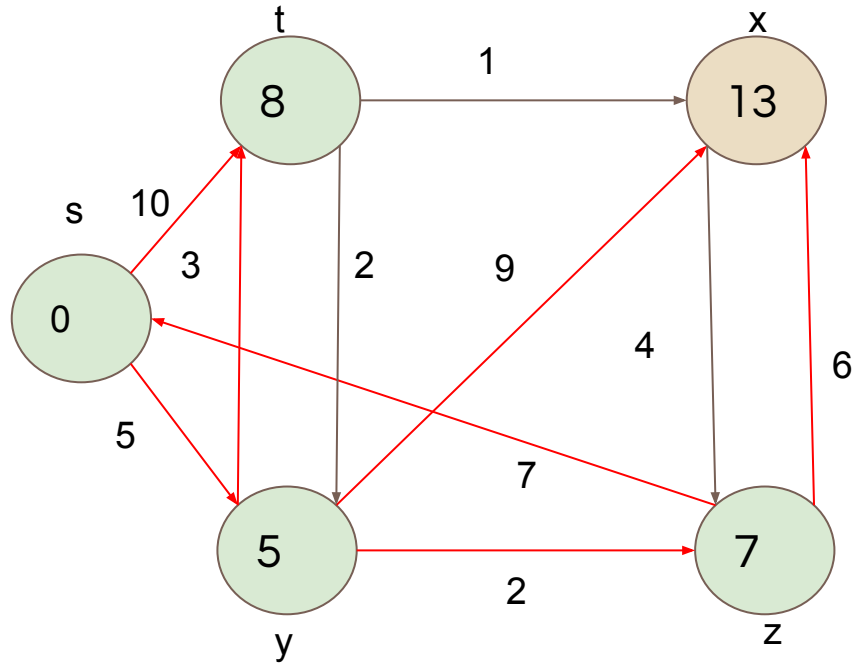
Dijkstra's algorithm



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Relax all of its edges

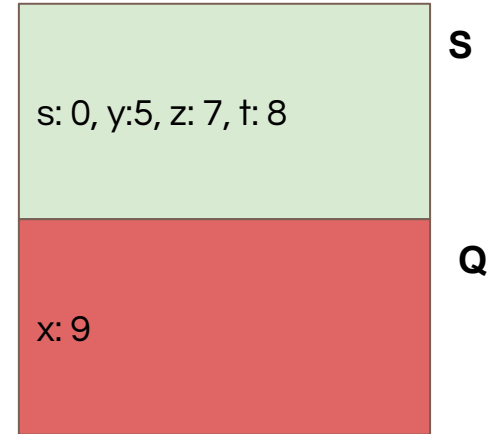
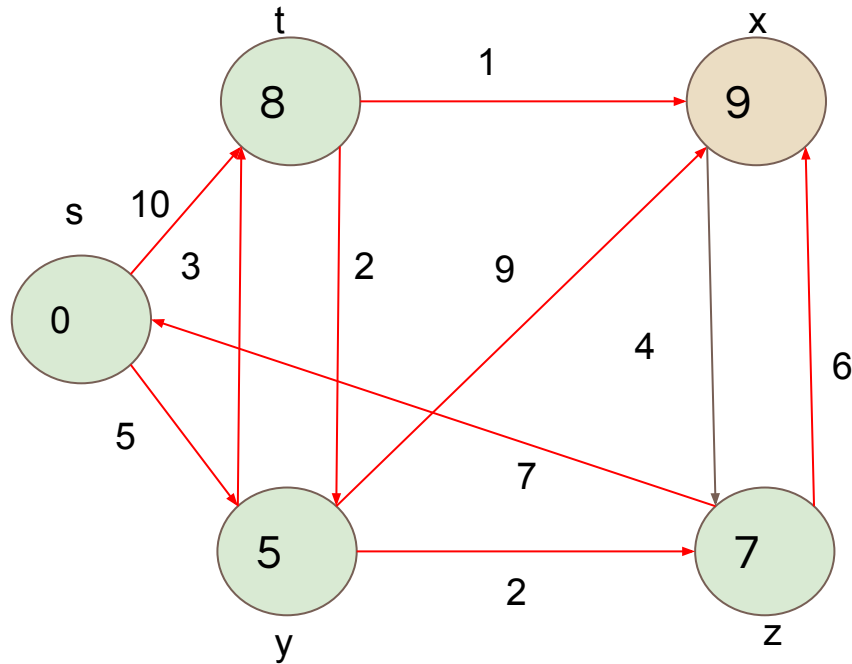
Dijkstra's algorithm



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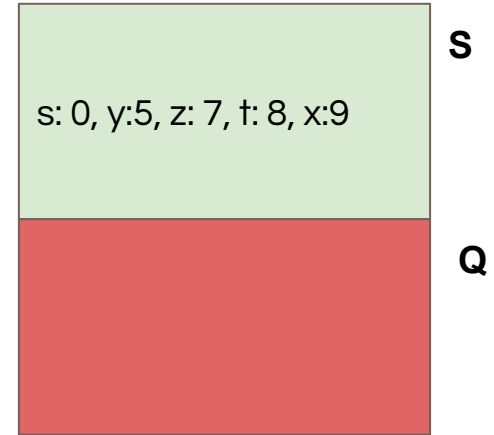
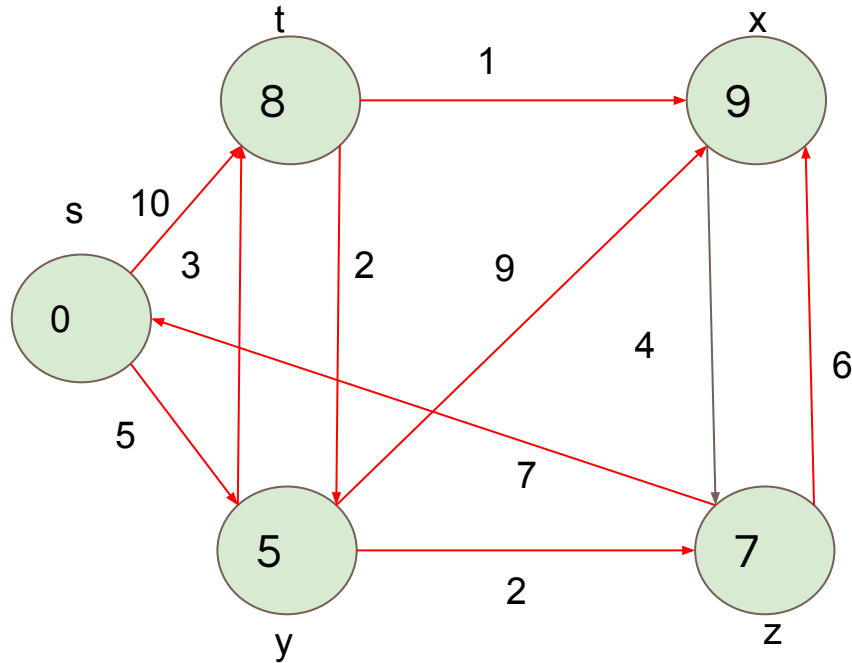
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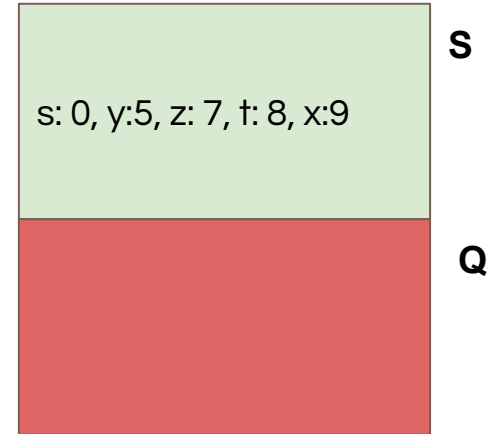
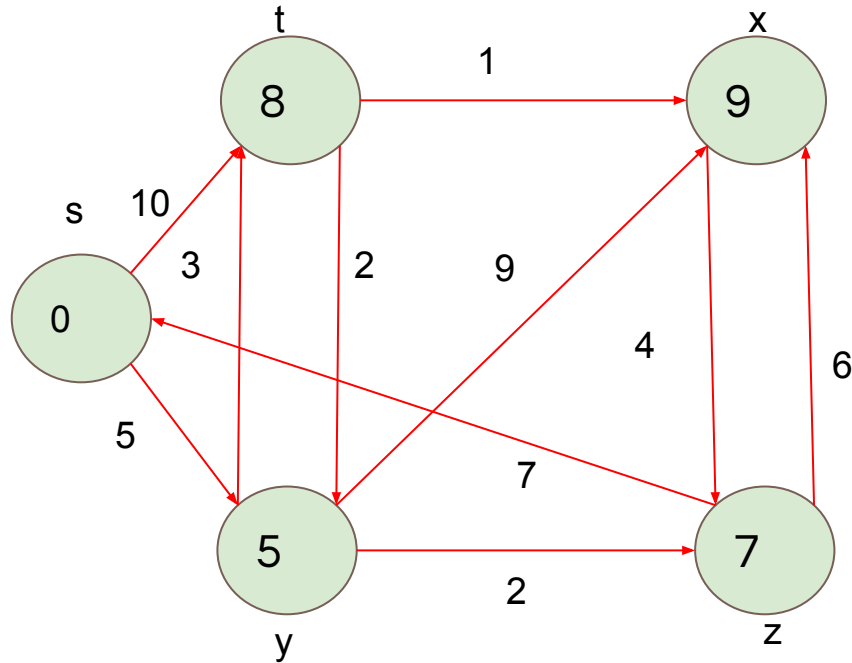
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Relax all of its edges

Dijkstra's algorithm



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Relax all of its edges

Dijkstra's algorithm

```
d[s,s] = 0
For v in V:
    d[s,v] = ∞
    π[v] = null
S = ∅
Q = V
while Q ≠ ∅
    u = FindMinimum from Q
    S = S ∪ {u}
    For each neighbour n of u:
        Relax(u,n)
```

```
Relax(u,n):
    If d[n] > d[u] + w(u,n):
        // Have discovered a shorter path
        d[n] = d[u] + w(u,n)
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Loop runs $O(V)$
times

At most relax
 $O(E)$ times

```
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$Q = V$

while $Q \neq \emptyset$

$u = \text{FindMinimum from } Q$

$S = S \cup \{u\}$

For each neighbour n of u :

Relax(u,n)

Call insert into Q
 $O(V)$ times

Call
FindMinimum
 $O(V)$ times

Call Relax $O(E)$
times.

Relax(u,n):

If $d[n] > d[u] + w(u,n)$:

// Have discovered a shorter path

$d[n] = d[u] + w(u,n)$

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Complexity - Priority Queue!

$d[s,s] = 0$

For v in V :

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$S = \emptyset$

$Q = \text{Insert}(V,Q)$

while $Q \neq \emptyset$

$u = \text{Extract-Min}(Q)$

$S = S \cup \{u\}$

For each neighbour n of u :

Relax(u,n)

Call insert into Q
 $O(V)$ times

Call Extract-Min
 $O(V)$ times

Call
DecreaseKey
 $O(E)$ times.

Relax(u,n):

If $d[n] > d[u] + w(u,n)$:

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Insert(v,Q): $O(\lg V)$

Extract-Min: $O(\lg V)$

Decrease-Key: $O(\lg V)$

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$O(V * \lg V + V * \lg V + E * \lg(V))$

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$d[n] = d[u] + w(u,n)$

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$\pi[n] = u$

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$O(V * \lg V + V * \lg V + E * \lg(V)) \Rightarrow O(V * \lg V + V * \lg V + E * O(1))$ if use **Fibonacci Heaps**

Optimal Substructure

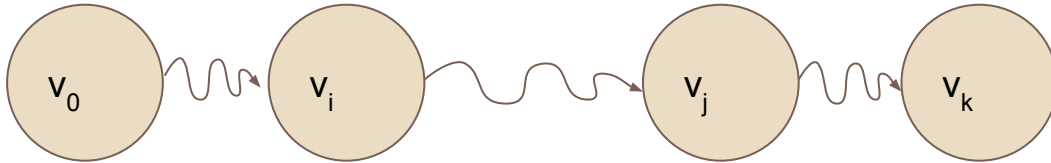
- Most shortest path algorithms rely on the **optimal substructure property**
 - Intuitively, says that **a shortest path between two vertices contains only other shortest paths within it**
 - If path $p = (v_0, v_1, v_2)$ from v_0 to v_2 is the shortest path from v_0 to v_2 , then (v_0, v_1) must also be the shortest path from v_0 to v_1 . Otherwise there'd be a better way to get to v_2 !

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- Given a graph $G=(V,E,W)$, let $p = (v_0, v_1, \dots, v_k)$ be a shortest path from vertex v_0 to vertex v_k and for any i and j such that $0 \leq i < j \leq k$, let p_{ij} be the subpath of p from vertex v_i to vertex v_j . Then p_{ij} is the shortest path from v_i to v_j .

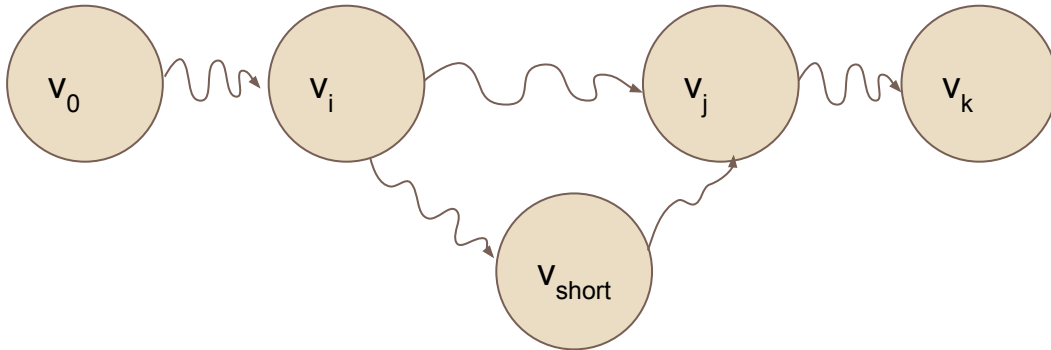
Optimal Substructure

- Proof by contradiction:
 - Assume that $p = (v_0, \dots, v_i, \dots, v_j, \dots, v_k)$ is the shortest path



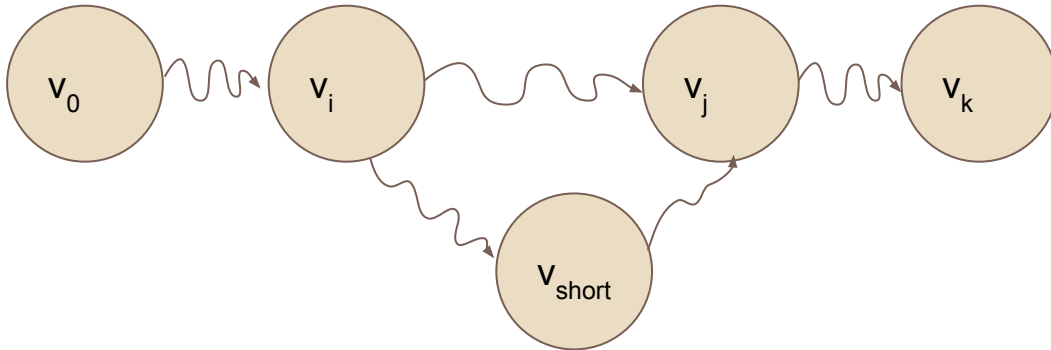
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 - Assume that there exists a shorter path between vertices i and vertices j .



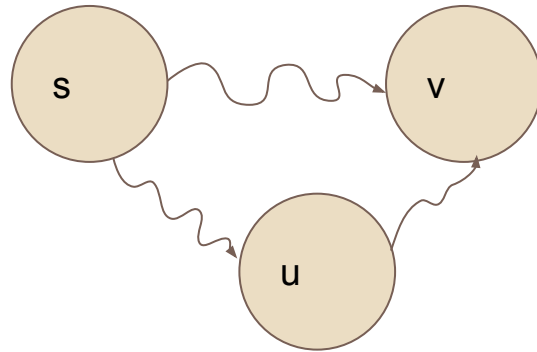
Optimal Substructure

- Proof by contradiction:
 - Assume that $p = (v_0, \dots, v_i, \dots, v_j, \dots, v_k)$ is the shortest path
 - Assume that there exists a shorter path between vertices i and vertices j .
 - Then the shortest path from v_0 to v_k would be via v_{short} so p is not the shortest path. **We have a contradiction**



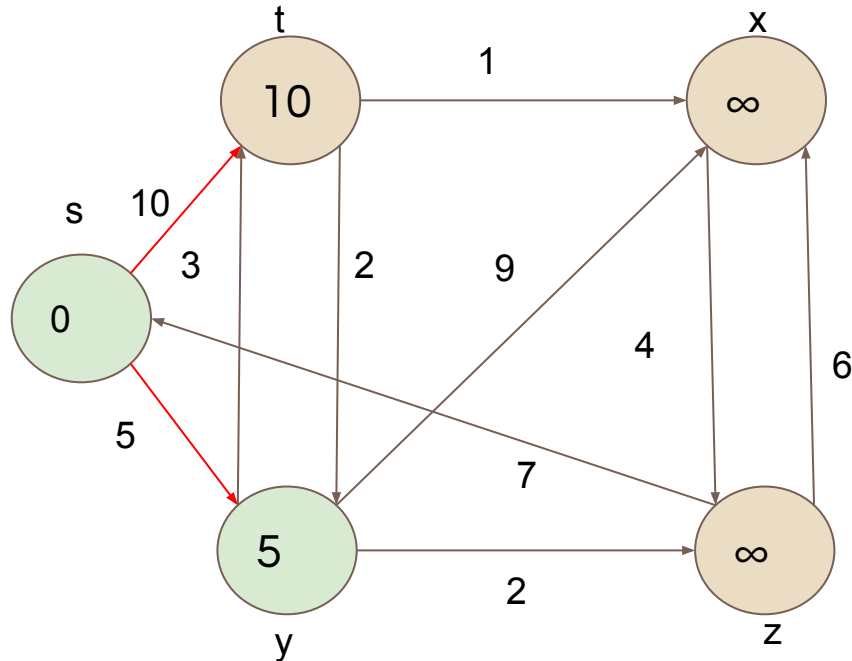
Triangle Inequality

- By the same logic, can derive the **triangle inequality**
 - $\delta(s,v) \leq \delta(s,u) + \delta(u,v)$



If the path (s .. v) is a shortest path, the weight of the path from (s,u) and from (u,v) cannot be smaller as that would mean that the path (s .. v) is not the shortest path

Dijkstra's algorithm - Again



Why is $d[s,y] = \delta(s,y)$?

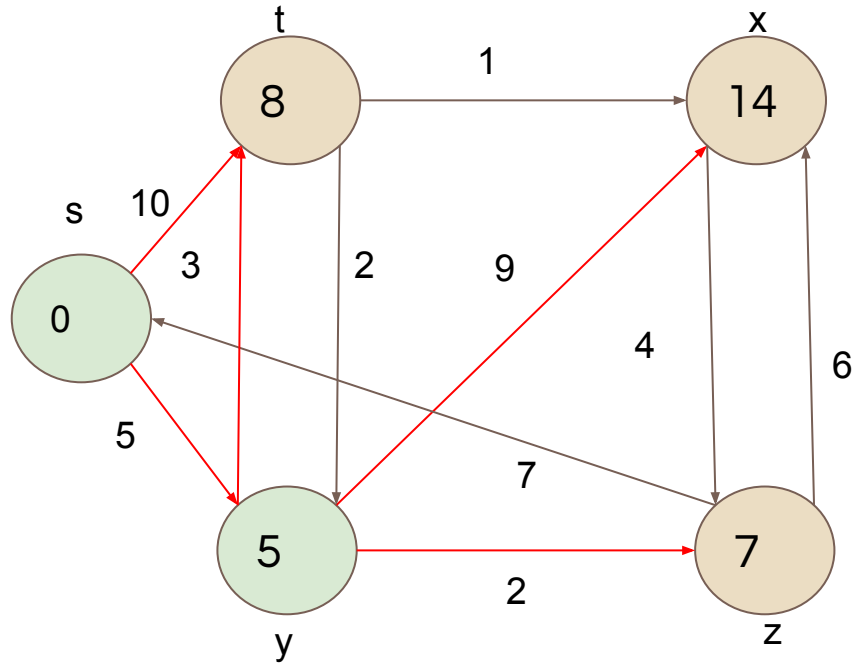
We have relaxed all the edges leaving s.

The only way to reach y is via (s,t) + (unknown path p) or via (s,y)

But $w(s,t) > w(s,y)$ so $w(s,t) + p > w(s,y)$ because $w(p) > 0$

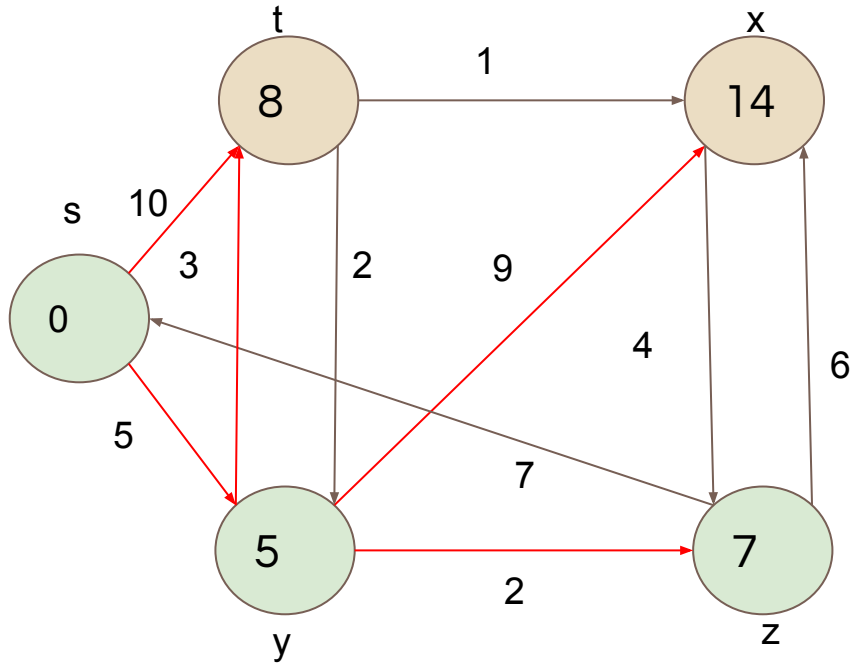
Any path that we take via t will have greater weight than $w(s,y)$, so $d[s,y] = \delta(s,y)$

Dijkstra's algorithm - Again



Now relax all of the edges that start from y, and update the current estimate of the shortest path.

Dijkstra's algorithm - Again



Why is $d[s,z] = \delta(s,z)$?

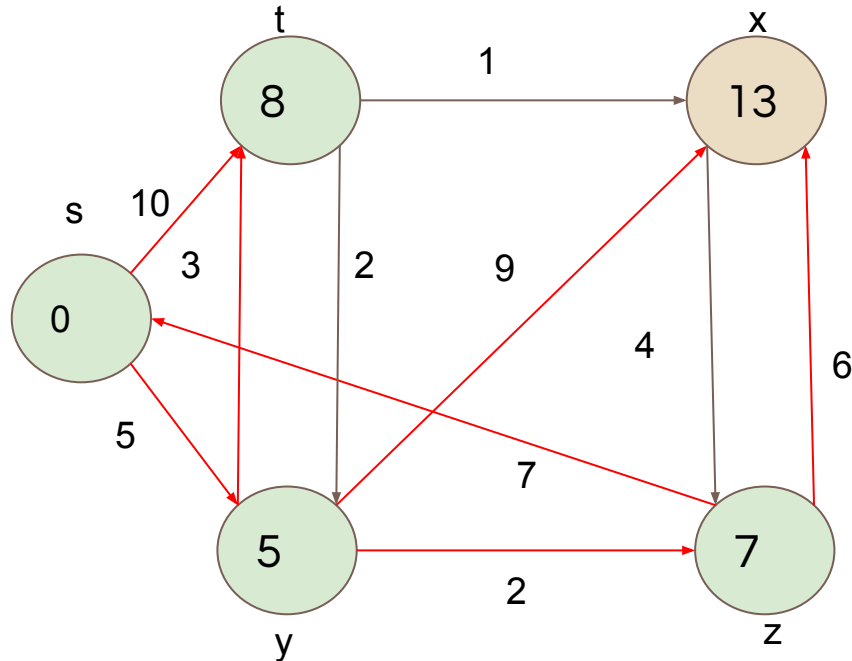
The current values represent **our best attempts to reach nodes t,x,z** using nodes s and y (because relaxed edges from s,y)

We want to show that reaching **z** through other nodes **t** and **x** would yield a value d that is greater than $d[z]$.

Going through s,y,x (...) z would not lead a shorter path as $d[s,x] = 14$

Going through s,y,t (...) z (the current shortest path to t) would not lead a shorter path as $d[s,t] = 8$

Dijkstra's algorithm - Again



Why is $d[s,t] = \delta(s,t)$?

The current values represent **our best attempts to reach nodes t,x** using nodes s,y,z (because relaxed edges from s,y,z)

We want to show that reaching **t** through other nodes **x** would yield a value d that is greater than $d[t]$.

Going through s,y,z,x (the current shortest path to x) would not lead a shorter path as $d[s,x] = 13$

Correctness Proof (Intuition)

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- **Lemma:** Initialising $d[s] = 0$ and $d[v] = \infty$ for all $v \in V - \{s\}$ establishes $d[v] \geq \delta(s,v)$ for all $v \in V$, and this invariant is maintained over any sequence of relaxation steps. **Upper Bound Property**
- **Proof:**
 - At initialisation $d[x] = \infty$ so $d[x] \geq \delta(u,x)$ for all $x \in V$
 - Assume, after i relaxation steps, that for all nodes $x \in V$, $d[x] \geq \delta(u,x)$. And consider relaxing edge (x,v) (the $(i+1)$ th relaxation step):
 - If we relax (x,v) : $d[v] = d[x] + w(x,v)$
 - By assumption $d[x] \geq \delta(u,x)$
 - It follows that $d[v] \geq \delta(u,x) + w(x,v)$.
 - It follows that $d[v] \geq \delta(u,x) + \delta(x,v)$. By definition, $w(x,v) \geq \delta(x,v)$
 - It follows that $d[v] \geq \delta(u,x) + \delta(x,v) \geq \delta(u,v)$ (by triangle inequality)

Correctness Proof (Intuition)

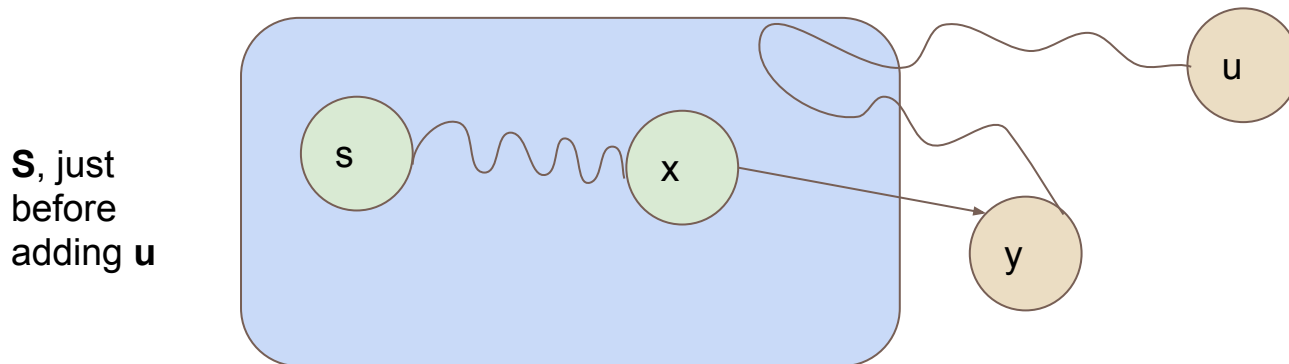
- **Theorem:** Dijkstra's algorithm terminates with $d[v]=\delta(s,v)$ for all in $v \in V$
- **Proof:** Want to show that $d[v]=\delta(s,v)$ for every $v \in V$ when v is added to **S**

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- **Theorem:** Dijkstra's algorithm terminates with $d[v]=\delta(s,v)$ for all $v \in V$
- **Proof:** Want to show that $d[v]=\delta(s,v)$ for every $v \in V$ when v is added to **S**
 - Suppose u is the first vertex added to **S** for which $d[u] \neq \delta(s,u)$
 - Let y be the first vertex in **Q** along a shortest path from s to u , and let x be its **predecessor**

Correctness Proof (Intuition)

- Since u is the first vertex violating the invariant, we have $d[x] = \delta(s,x)$
- Since subpaths of shortest paths are shortest paths, and y is on shortest path from s to u , $d[y]$ was set to $\delta(s,x) + w(x,y) = \delta(s,y)$ just after x was added to S
- We have $d[y] = \delta(s,y)$ and $\delta(s,y) \leq \delta(s,u) \leq d[u]$ (Upper Bound Property)



Correctness Proof (Intuition)

- But, $d[y] \geq d[u]$ since the algorithm chose u first
- Hence $d[y] = \delta(s,y) = \delta(s,u) = d[u]$
- We have a contradiction! So $d[u] = \delta(s,u)$

S, just
before
adding **u**

