"Organizing is what you do before you do something, so that when you do it, it is not all mixed up."
~ A. A. Milne

# SORTING

Lecture 11
CS2110 – Fall 2017

## Prelim 1

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- It's on Tuesday Evening (3/13)
- Two Sessions:
  - 5:30-7:00PM: netid aa..ks
  - 7:30-9:00PM: netid kt..zz
  - If you have a conflict with your assigned time but can make the other time, fill out conflict assignment on CMS BY TOMORROW
- Three Rooms:
  - We will email you Tuesday morning with your room
- Bring your Cornell ID!!!

## Prelim 1

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- Recitation 5: prelim review
- Review Session: Sunday 3/11, 1-3pm in Kimball B11
- Study guide on course website

## Why Sorting?

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- Sorting is useful
  - Database indexing
  - Operations research
  - Compression
- There are lots of ways to sort
  - There isn't one right answer
  - You need to be able to figure out the options and decide which one is right for your application.
  - Today, we'll learn about several different algorithms (and how to derive them)

## Some Sorting Algorithms

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- Insertion sort
- Selection sort
- Merge sort
- Quick sort

## InsertionSort

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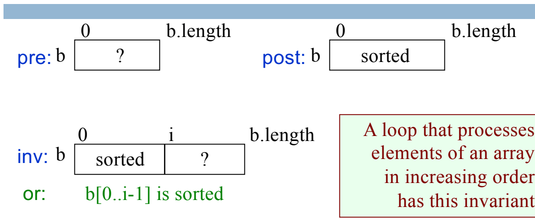

## Each iteration, i= i+1; How to keep inv true?

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| | | 0 | i | b.length |
|---|---|---|---|---|
| inv: | b | sorted | ? | |
| e.g. | b | 2 5 5 5 7 | 3 ? | |
| | b | 2 3 5 5 5 7 | ? | |

## What to do in each iteration?

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| | | 0 | i | b.length |
|---|---|---|---|---|
| inv: | b | sorted | ? | |
| e.g. | b | 2 5 5 5 7 | 3 ? | |

Loop body (inv true before and after):

| | |
|---|---|
| 2 5 5 5 3 | 7 ? |
| 2 5 5 3 5 | 7 ? |
| 2 5 3 5 5 | 7 ? |
| 2 3 5 5 5 | 7 ? |

Push b[i] to its sorted position in b[0..i], then increase i

This will take time proportional to the number of swaps needed

b | 2 3 5 5 5 7 | ?

## Insertion Sort

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```
// sort b[], an array of int
// inv: b[0..i-1] is sorted
for (int i= 0; i < b.length; i= i+1) {
   // Push b[i] down to its sorted
   // position in b[0..i]
}
```

Present algorithm like this

Note English statement in body. **Abstraction**. Says **what** to do, not **how.**

This is the best way to present it. We expect you to present it this way when asked.

Later, can show how to implement that with an inner loop

## Insertion Sort

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```
// sort b[], an array of int
// inv: b[0..i-1] is sorted
for (int i= 0; i < b.length; i= i+1) {
   // Push b[i] down to its sorted
   // position in b[0..i]
   int k= i;
   while (k > 0 && b[k] < b[k-1]) {
      Swap b[k] and b[k-1]
      k= k–1;
   }
}
```

invariant P: b[0..i] is sorted **except** that b[k] may be < b[k-1]

| | k | | i | |
|---|---|---|---|---|
| 2 5 | 3 | 5 5 | 7 | ? |

example

start?

stop?

progress?

maintain invariant?

## Insertion Sort

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```
// sort b[], an array of int
// inv: b[0..i-1] is sorted
for (int i= 0; i < b.length; i= i+1) {
   // Push b[i] down to its sorted
   // position in b[0..i]}
```

Pushing b[i] down can take i swaps. Worst case takes

1 + 2 + 3 + … n-1 = (n-1)*n/2

Swaps.

Let n = b.length

- Worst-case: $O(n^2)$ (reverse-sorted input)
- Best-case: $O(n)$ (sorted input)
- Expected case: $O(n^2)$

## Performance

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| Algorithm | Time | Space | Stable? |
|---|---|---|---|
| Insertion Sort | $O(n)$ to $O(n^2)$ | $O(1)$ | Yes |
| Selection Sort | $O(n^2)$ | $O(1)$ | No |
| Merge Sort | | | |
| Quick Sort | | | |

## SelectionSort

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pre: b [0 ? b.length]   post: b [0 sorted b.length]

inv: b [0 sorted , <= b[i..] i >= b[0..i-1] b.length]

Additional term in invariant

Keep invariant true while making progress?

e.g.: b [0 1 2 3 4 5 6 i 9 9 9 7 8 6 9 b.length]

Increasing i by 1 keeps inv true only if b[i] is min of b[i..]

## SelectionSort

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```
//sort b[], an array of int
// inv: b[0..i-1] sorted AND
//      b[0..i-1] <= b[i..]
for (int i= 0; i < b.length; i= i+1) {
  int m= index of minimum of b[i..];
  Swap b[i] and b[m];
}
```

Another common way for people to sort cards

Runtime with n = b.length

- Worst-case $O(n^2)$
- Best-case $O(n^2)$
- Expected-case $O(n^2)$

b [0 sorted, smaller values i larger values length]

Each iteration, swap min value of this section into b[i]

## Performance

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| Algorithm | Time | Space | Stable? |
|---|---|---|---|
| Insertion Sort | $O(n)$ to $O(n^2)$ | $O(1)$ | Yes |
| Selection Sort | $O(n^2)$ | $O(1)$ | No |
| Merge Sort | | | |
| Quick Sort | | | |

## Merge two adjacent sorted segments

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```
/* Sort b[h..k]. Precondition: b[h..t] and b[t+1..k] are sorted. */
public static merge(int[] b, int h, int t, int k) {
}
```

b [h 4 7 7 8 9 t 3 4 7 8 k]   [h sorted t sorted k]

↓

b [3 4 4 7 7 7 8 8 9]   [h merged, sorted k]

## Merge two adjacent sorted segments

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```
/* Sort b[h..k]. Precondition: b[h..t] and b[t+1..k] are sorted. */
public static merge(int[] b, int h, int t, int k) {
    Copy b[h..t] into a new array c;
    Merge c and b[t+1..k] into b[h..k];
}
```

[h sorted t sorted k]

↓

[h merged, sorted k]

## Merge two adjacent sorted segments

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// Merge sorted c and b[t+1..k] into b[h..k]

pre: c [0 x t-h]   b [h ? t y k]   x, y are sorted

post: b [h x and y, sorted k]

invariant: c [0 head of x i tail of x c.length]

b [h u ? v tail of y k]

head of x and head of y, sorted

## Merge

```
int i = 0;
int u = h;
int v = t+1;
while( i <= t-h){
    if(v < k && b[v] < c[i]) {
        b[u] = b[v];
        u++; v++;
    }else {
        b[u] = c[i];
        u++; i++;
    }
  }
}
```

pre: c [0 .. t-h: sorted]  b [h: ? | t: sorted .. k]

post: b [h .. k: sorted]

inv: c [0: sorted | i: sorted .. c.length]

b [h: sorted | u: ? | v: sorted .. k]

## Mergesort

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```
/** Sort b[h..k] */
public static void mergesort(int[] b, int h, int k]) {
    if (size b[h..k] < 2)
        return;
    int t= (h+k)/2;
    mergesort(b, h, t);
    mergesort(b, t+1, k);
    merge(b, h, t, k);
}
```

h [sorted | t sorted] k

h [merged, sorted] k

## Performance

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| Algorithm | Time | Space | Stable? |
|---|---|---|---|
| Insertion Sort | $O(n)$ to $O(n^2)$ | $O(1)$ | Yes |
| Selection Sort | $O(n^2)$ | $O(1)$ | No |
| Merge Sort | $n\log(n)$ | $O(n)$ | Yes |
| Quick Sort | | | |

## QuickSort

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Quicksort developed by Sir Tony Hoare (he was knighted by the Queen of England for his contributions to education and CS).

83 years old.

Developed Quicksort in 1958. But he could not explain it to his colleague, so he gave up on it.

Later, he saw a draft of the new language Algol 58 (which became Algol 60). It had recursive procedures. First time in a procedural programming language. "Ah!," he said. "I know how to write it better now." 15 minutes later, his colleague also understood it.

## Partition algorithm of quicksort

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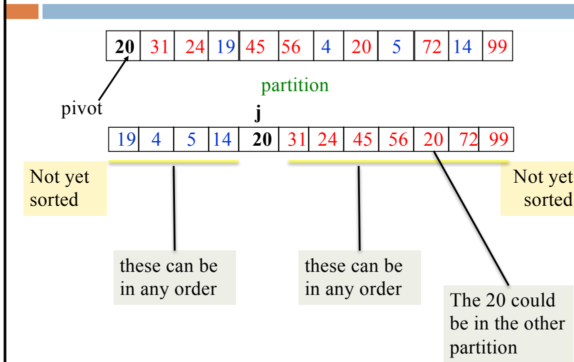
pre: h [x | h+1 ? ] k

x is called the pivot

Swap array values around until b[h..k] looks like this:

post: h [<= x | j x | >= x] k

## Partition algorithm of quicksort



## Partition algorithm

pre: b

| h | h+1 | k |
|---|---|---|
| x | ? | |

post: b

| h | j | k |
|---|---|---|
| <= x | x | >= x |

Combine pre and post to get an invariant

b

| h | j | t | k |
|---|---|---|---|
| <= x | x | ? | >= x |

invariant needs at least 4 sections

## Partition algorithm

b

| h | j | t | k |
|---|---|---|---|
| <= x | x | ? | >= x |

Initially, with j = h and t = k, this diagram looks like the start diagram

```
j= h; t= k;
while (j < t) {
   if (b[j+1] <= b[j]) {
      Swap b[j+1] and b[j];   j= j+1;
   } else {
      Swap b[j+1] and b[t];   t= t-1;
   }
}
```

Terminate when j = t, so the "?" segment is empty, so diagram looks like result diagram

Takes linear time: O(k+1-h)

## QuickSort procedure

```
/** Sort b[h..k]. */
public static void QS(int[] b, int h, int k) {
   if (b[h..k] has < 2 elements) return;
   int j= partition(b, h, k);
      // We know b[h..j–1] <= b[j] <= b[j+1..k]
      // Sort b[h..j-1] and b[j+1..k]
      QS(b, h, j-1);
      QS(b, j+1, k);
}
```

Base case

Function does the partition algorithm and returns position j of pivot

| h | j | k |
|---|---|---|
| <= x | x | >= x |

## Worst case quicksort: pivot always smallest value

| j | | n |
|---|---|---|
| x0 | >= x0 | |

partioning at depth 0

| | j | |
|---|---|---|
| x0 | x1 | >= x1 |

partioning at depth 1

| | | j | |
|---|---|---|---|
| x0 | x1 | x2 | >= x2 |

partioning at depth 2

```
/** Sort b[h..k]. */
public static void QS(int[] b, int h, int k) {
   if (b[h..k] has < 2 elements) return;
   int j= partition(b, h, k);
   QS(b, h, j-1);     QS(b, j+1, k);
```

Depth of recursion: O(n)

Processing at depth i: O(n-i)

O(n*n)

## Best case quicksort: pivot always middle value

| 0 | j | n |
|---|---|---|
| <= x0 | x0 | >= x0 |

depth 0. 1 segment of size ~n to partition.

| <=x1 | x1 | >= x1 | x0 | <=x2 | x2 | >=x2 |
|---|---|---|---|---|---|---|

Depth 2. 2 segments of size ~n/2 to partition.

Depth 3. 4 segments of size ~n/4 to partition.

Max depth: O(log n). Time to partition on each level: O(n)
Total time: O(n log n).

Average time for Quicksort: n log n. Difficult calculation

## QuickSort complexity to sort array of length n

Time complexity
Worst-case: O(n*n)
Average-case: O(n log n)

```
/** Sort b[h..k]. */
public static void QS(int[] b, int h, int k) {
   if (b[h..k] has < 2 elements) return;
   int j= partition(b, h, k);
   // We know b[h..j–1] <= b[j] <= b[j+1..k]
   // Sort b[h..j-1] and b[j+1..k]
   QS(b, h, j-1);
   QS(b, j+1, k);
}
```

Worst-case space: ?
What's depth of recursion?

Worst-case space: O(n)!
--depth of recursion can be n
Can rewrite it to have space O(log n)
Show this at end of lecture if we have time

## QuickSort versus MergeSort

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```
/** Sort b[h..k] */
public static void QS
      (int[] b, int h, int k) {
   if (k – h < 1) return;
   int j= partition(b, h, k);
   QS(b, h, j-1);
   QS(b, j+1, k);
}
```

```
/** Sort b[h..k] */
public static void MS
      (int[] b, int h, int k) {
   if (k  – h < 1) return;
   MS(b, h, (h+k)/2);
   MS(b, (h+k)/2 + 1, k);
   merge(b, h, (h+k)/2, k);
}
```

One processes the array then recurses.
One recurses then processes the array.

## Partition. Key issue. How to choose pivot

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pre: b [ x | ? ] (h h … k)

post: b [ <= x | x | >= x ] (h … j … k)

Choosing pivot
Ideal pivot: the median, since it splits array in half
But computing is O(n), quite complicated

Popular heuristics: Use
- first array value (not so good)
- middle array value (not so good)
- Choose a random element (not so good)
- median of first, middle, last, values (often used)!

## Performance

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| Algorithm | Time | Space | Stable? |
|---|---|---|---|
| Insertion Sort | $O(n)$ to $O(n^2)$ | $O(1)$ | Yes |
| Selection Sort | $O(n^2)$ | $O(1)$ | No |
| Merge Sort | $n\log(n)$ | $O(n)$ | Yes |
| Quick Sort | $n\log(n)$ to $O(n^2)$ | $O(\log(n))$ | No |

## Sorting in Java

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- Java.util.Arrays has a method Sort()
  - implemented as a collection of overloaded methods
  - for primitives, Sort is implemented with a version of quicksort
  - for Objects that implement Comparable, Sort is implemented with mergesort
- Tradeoff between speed/space and stability/performance guarantees

## Quicksort with logarithmic space

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Problem is that if the pivot value is always the smallest (or always the largest), the depth of recursion is the size of the array to sort.

Eliminate this problem by doing some of it iteratively and some recursively. We may show you this later. Not today!

## QuickSort with logarithmic space

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```
/** Sort b[h..k]. */
public static void QS(int[] b, int h, int k) {
   int h1= h; int k1= k;
   // invariant b[h..k] is sorted if b[h1..k1] is sorted
   while (b[h1..k1] has more than 1 element) {
        Reduce the size of b[h1..k1], keeping inv true
   }
}
```

## QuickSort with logarithmic space

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```
/** Sort b[h..k]. */
public static void QS(int[] b, int h, int k) {
   int h1= h; int k1= k;
   // invariant b[h..k] is sorted if b[h1..k1] is sorted
   while (b[h1..k1] has more than 1 element) {
        int j= partition(b, h1, k1);
        // b[h1..j-1] <= b[j] <= b[j+1..k1]
        if (b[h1..j-1] smaller than b[j+1..k1])
             {  QS(b, h, j-1);  h1=  j+1; }
        else
             {QS(b, j+1, k1);  k1=  j-1; }
   }
}
```

Only the smaller segment is sorted recursively. If b[h1..k1] has size n, the smaller segment has size < n/2. Therefore, depth of recursion is at most log n