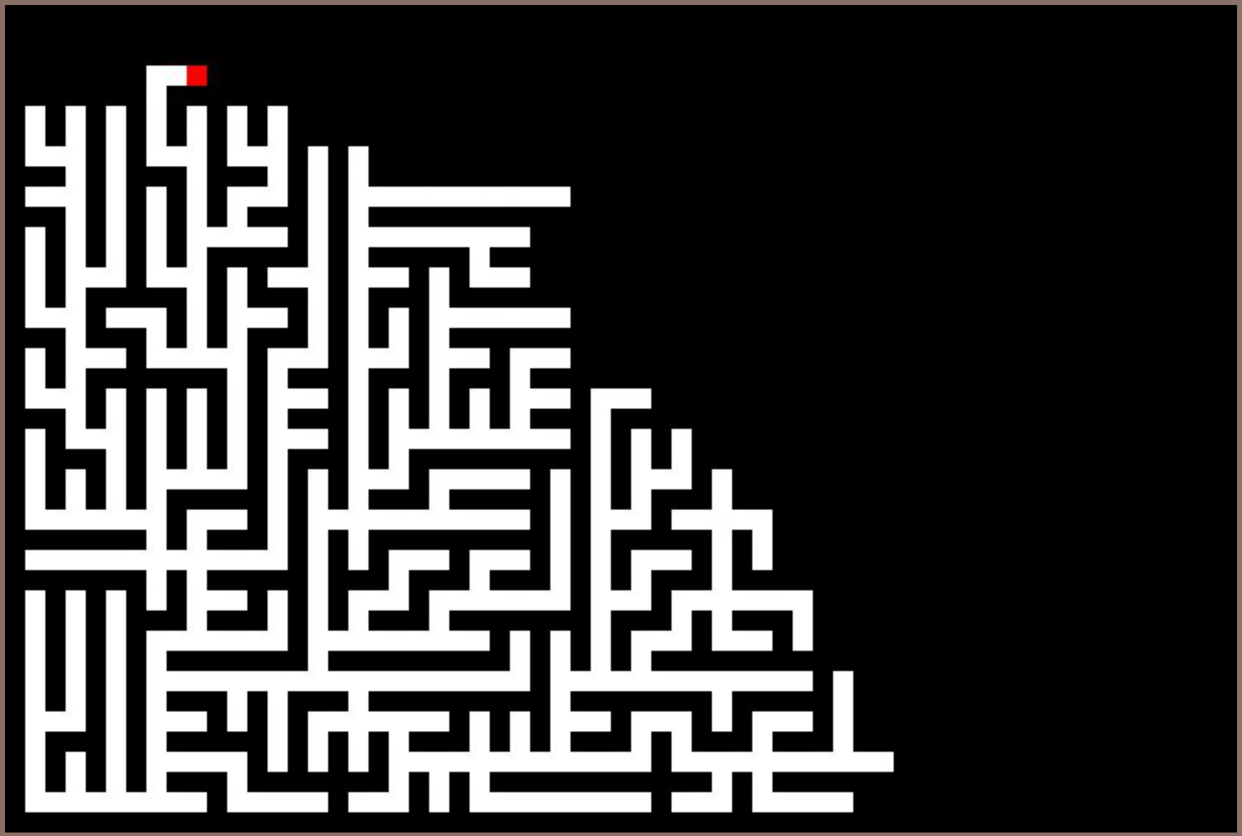


SPANNING TREES

Lecture 21

CS2110 – Fall 2016

Spanning trees

What we do today:

- Talk about modifying an existing algorithm
- Calculating the shortest path in Dijkstra's algorithm
- Minimum spanning trees
- 3 greedy algorithms (including Kruskal & Prim)

Assignment A7 available soon
Due close to prelim 2

Implement Dijkstra's shortest-path algorithm.

Start with our abstract algorithm, implement it in a specific setting. Our method: 36-40 lines, including extensive comments.

We give you all necessary test cases.

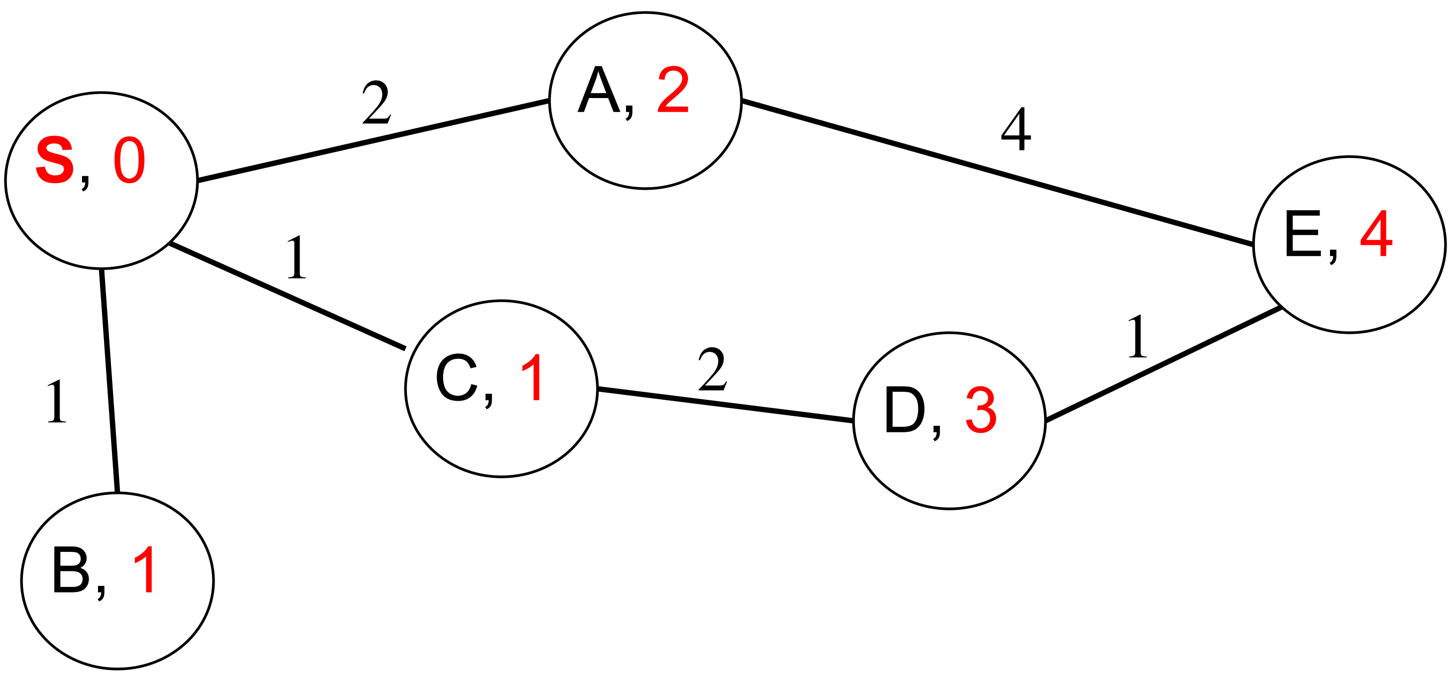
We will make our solution to A6 available after the deadline for late submissions.

Pxrevious semester: median: 4.0, mean: 3.84. But our abstract algorithm is much closer to the planned implementation than during that semester, and we expect a much lower median and mean.

Dijkstra's algorithm using class SFdata.

An object of class Sfdata for each node of the graph.

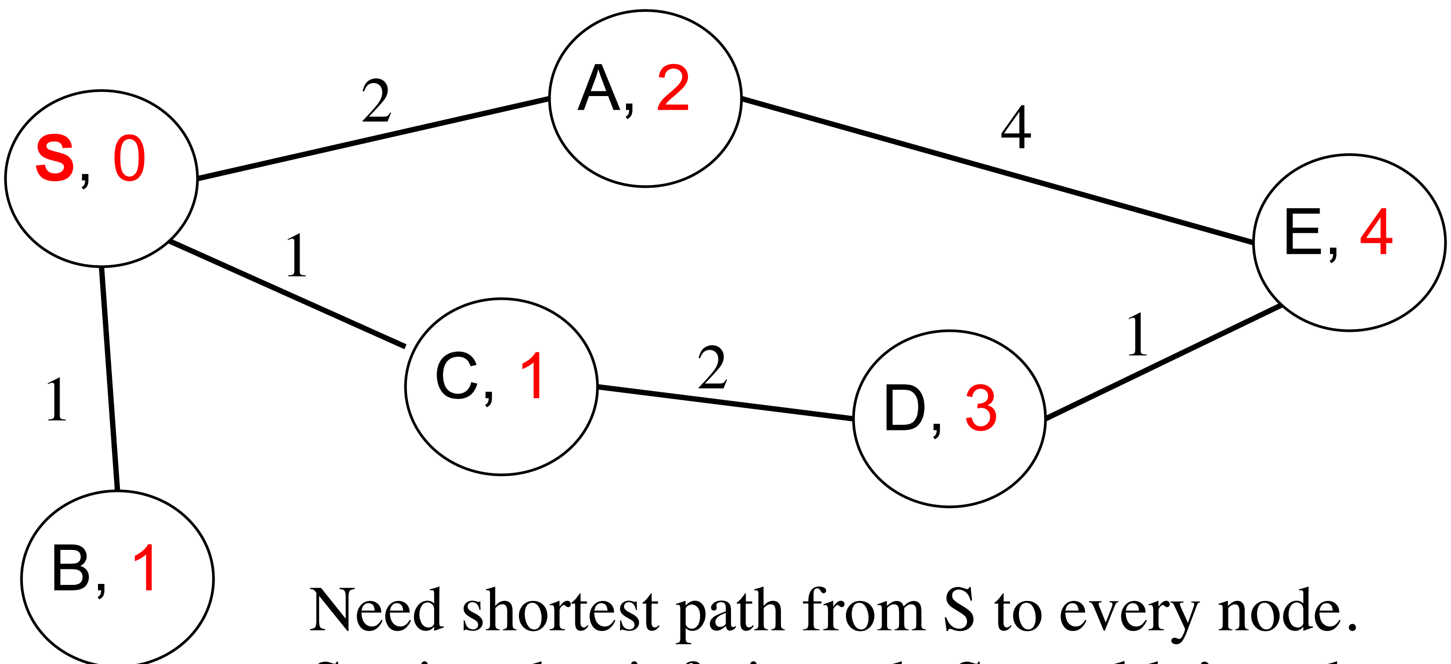
SFdata contains shortest distance from Start node (red).



Backpointers

Shortest path requires not only the distance from start to a node but the shortest path itself. How to do that?

In the graph, red numbers are shortest distance from S.

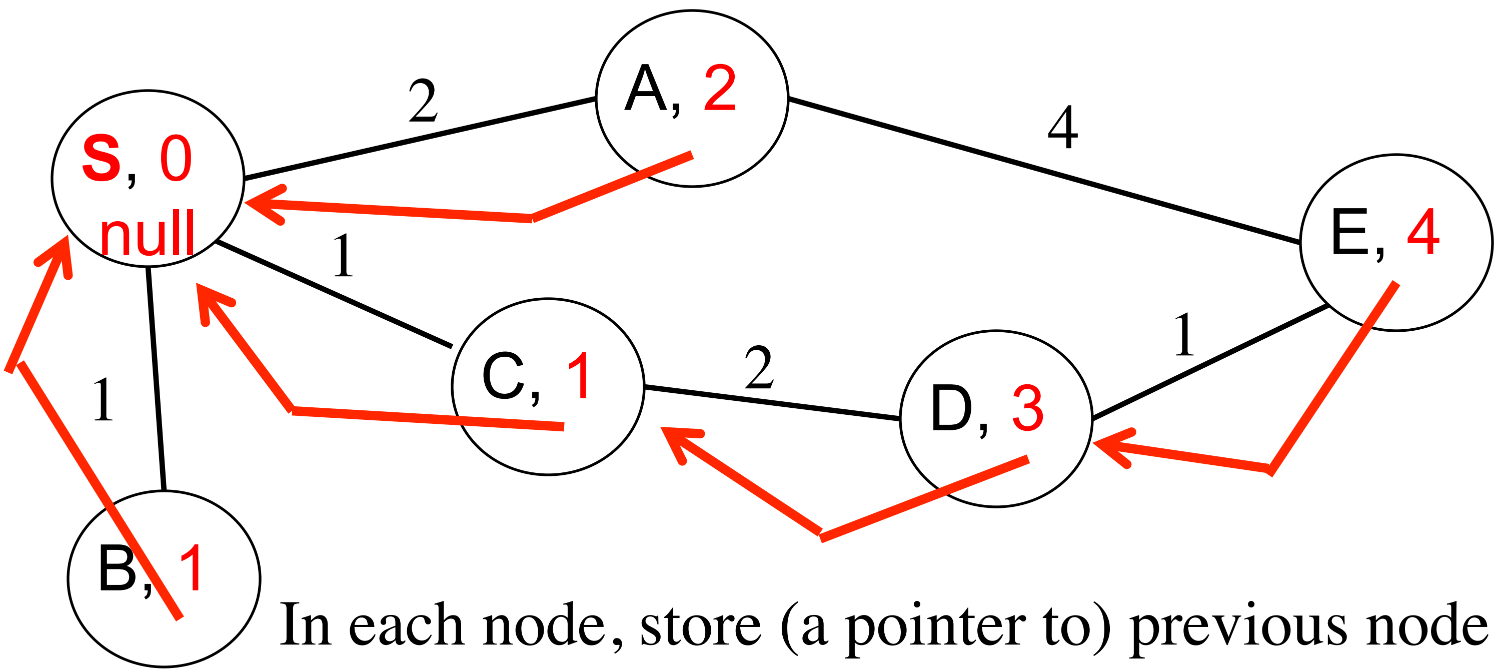


Need shortest path from S to every node.
Storing that info in node S wouldn't make sense.

Backpointers

Shortest path requires not only the distance from start to a node but the shortest path itself. How to do that?

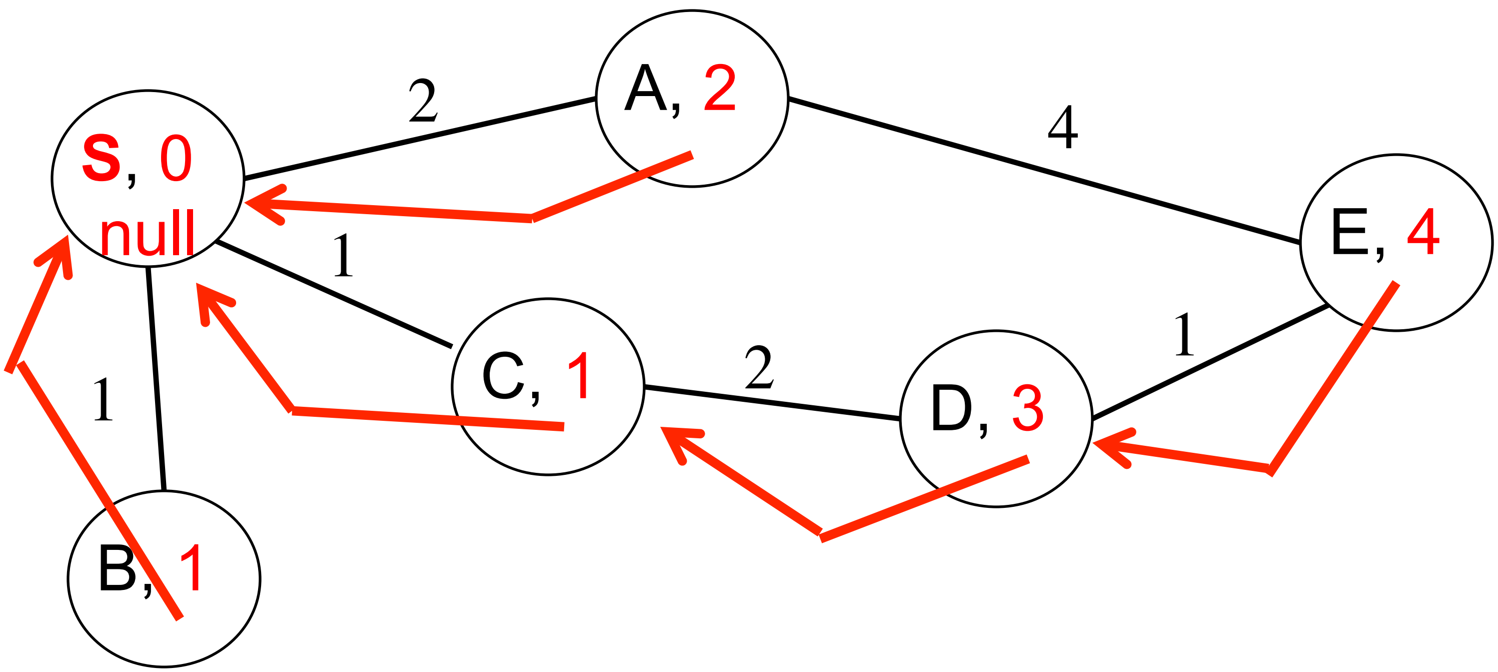
In the graph, red numbers are shortest distance from S.



In each node, store (a pointer to) previous node on the shortest path from S to that node. **Backpointer**

Backpointers

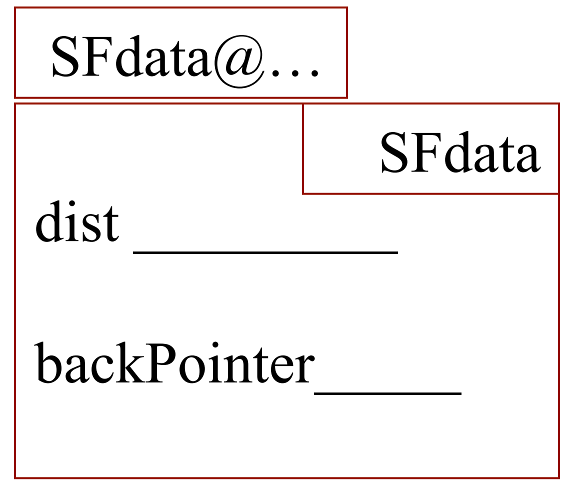
When to set a backpointer? In the algorithm, processing an edge (f, w): If the shortest distance to w changes, then set w's backpointer to f. It's that easy!



Each iteration of Dijkstra's algorithm

dist: shortest-path length calculated so far

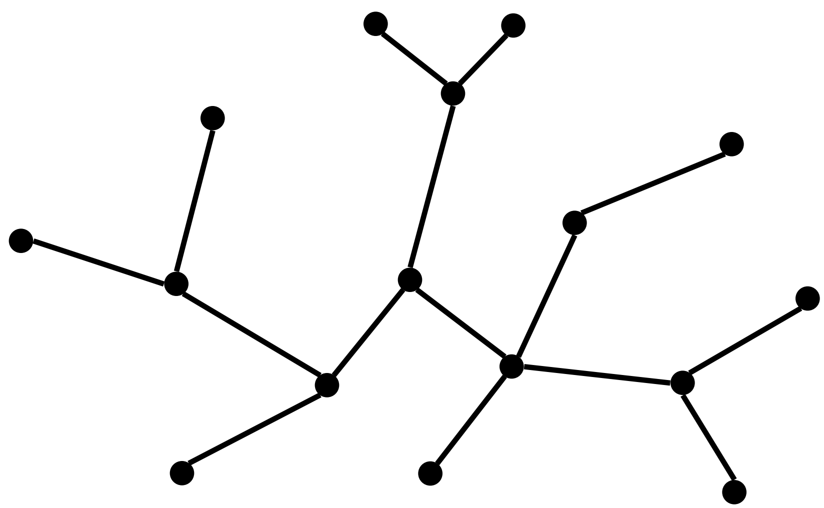

```
f= node in Frontier with min dist; Remove f from Frontier;
for each neighbor w of f:
    if w in far-off set
    then    w.spl=  f.dist + weight(f, w);
            Put w in the Frontier;
            w.backPointer=  f;
    else if f.dist + weight(f, w) < w.dist
         then   w.dist=  f.dist + weight(f, w);
                w.backPointer=  f;
```



Undirected trees

- An undirected graph is a *tree* if there is exactly one simple path between any pair of vertices

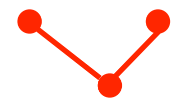
Root of tree?
It doesn't matter. Choose any vertex for the root

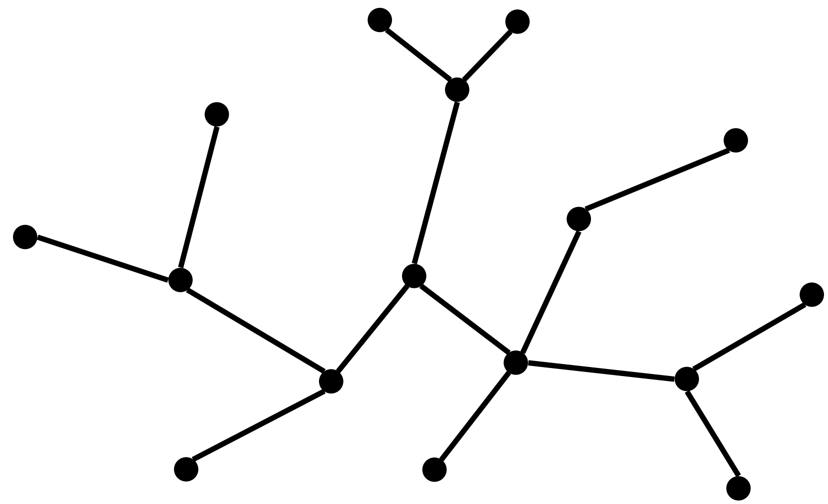


Facts about trees

Consider a graph with these properties:

1. $|E| = |V| - 1$
2. connected
3. no cycles



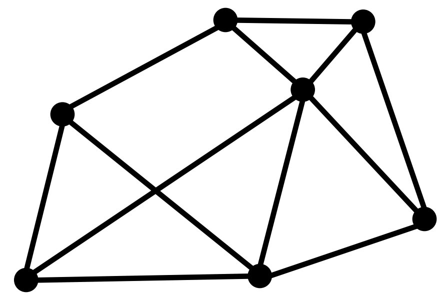


Any two of these properties imply the third —and imply that the graph is a tree

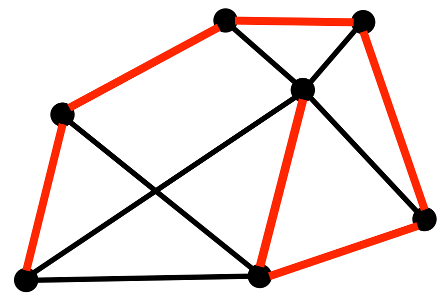
V: set of vertices
E: set of edges

A ***spanning tree*** of a **connected undirected** graph (V, E) is a subgraph (V, E') that is a tree

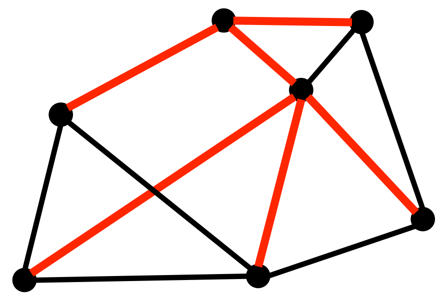
- Same set of vertices V
- E' ⊆ E
- (V, E') is a tree



- Same set of vertices V
- Maximal set of edges that contains no cycle

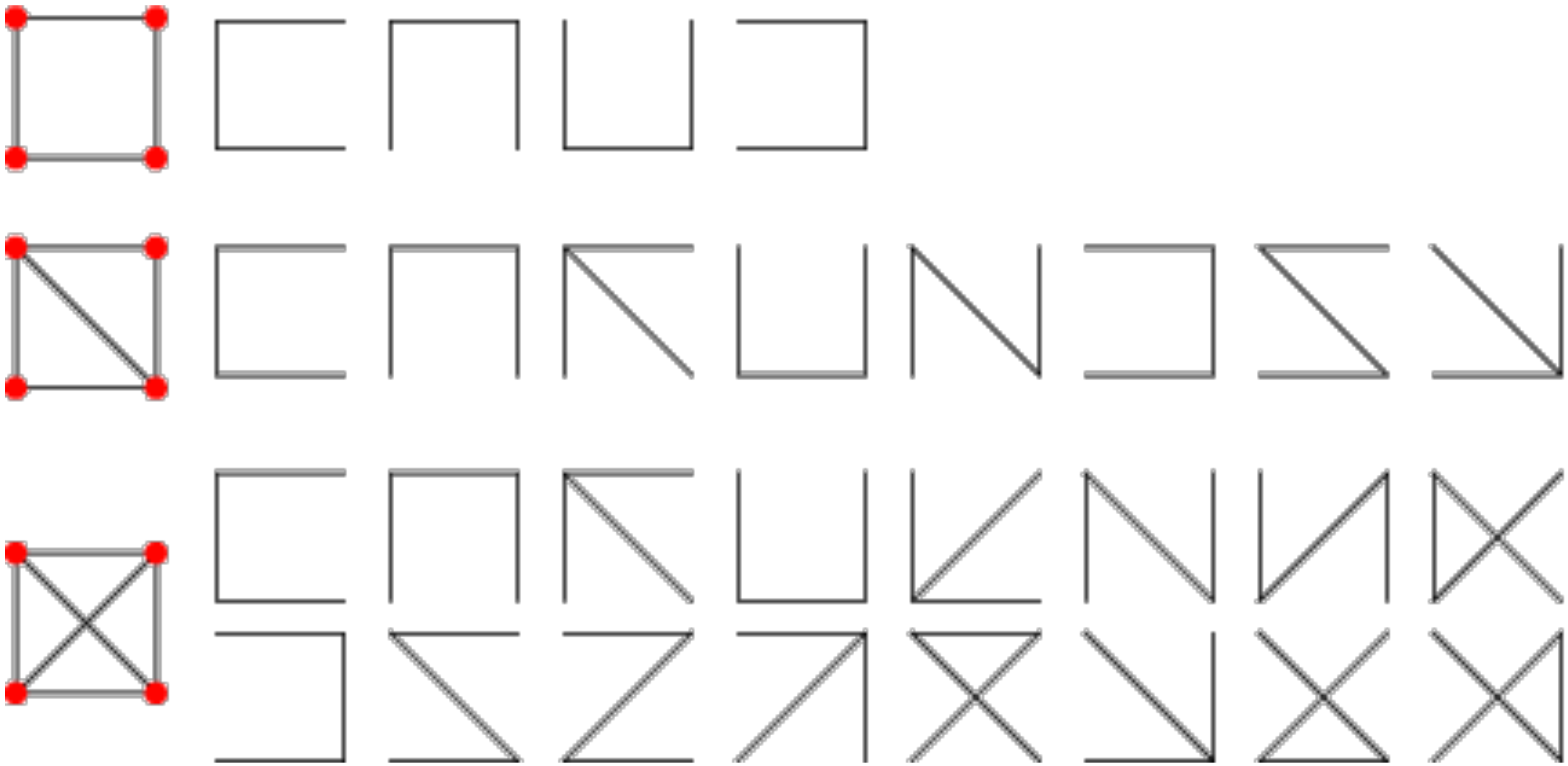


- Same set of vertices V
- Minimal set of edges that connect all vertices



Three equivalent definitions

Spanning trees: examples



http://mathworld.wolfram.com/SpanningTree.html

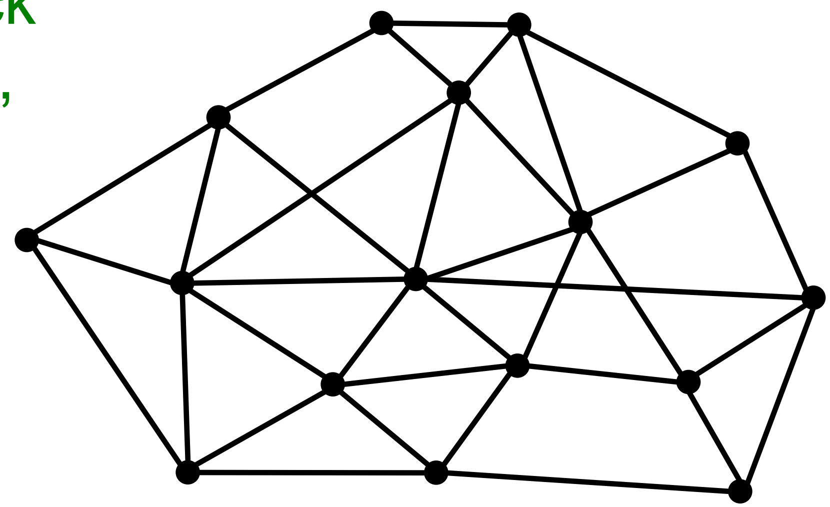
Finding a spanning tree

Use:

Maximal set of edges that contains no cycle

A subtractive method

- Start with the whole graph – it is connected
- If there is a cycle, pick an edge on the cycle, throw it out – the graph is still connected (why?)
- Repeat until no more cycles



Finding a spanning tree

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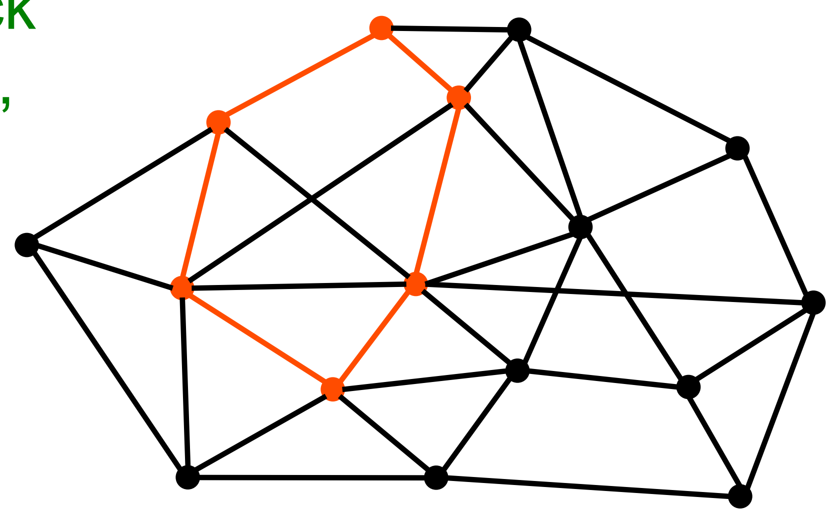

Finding a spanning tree

Use:

Maximal set of edges that contains no cycle

A subtractive method

- Start with the whole graph – it is connected
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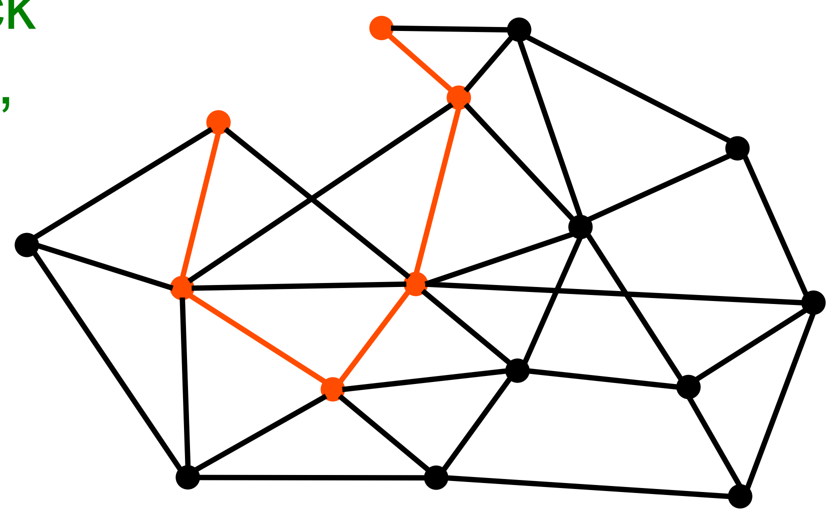


Nondeterministic algorithm

Finding a spanning tree: Additive method

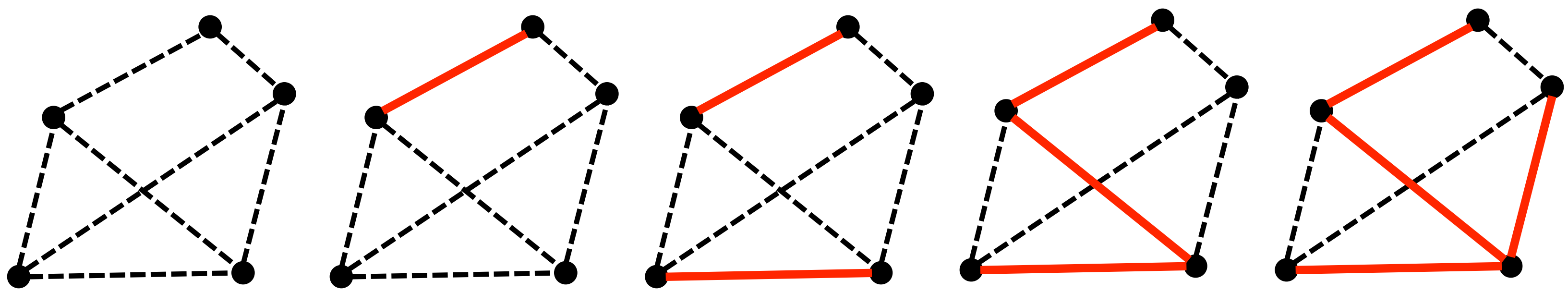
- Start with no edges
- While the graph is not connected:
  Choose an edge that connects 2 connected components and add it
  – the graph still has no cycle (why?)

Use:

Minimal set of edges that connects all vertices

nondeterministic algorithm

Tree edges will be red.
Dashed lines show original edges.
Left tree consists of 5 connected components, each a node



Minimum spanning trees

- Suppose edges are weighted (> 0), and we want a spanning tree of *minimum cost* (sum of edge weights)

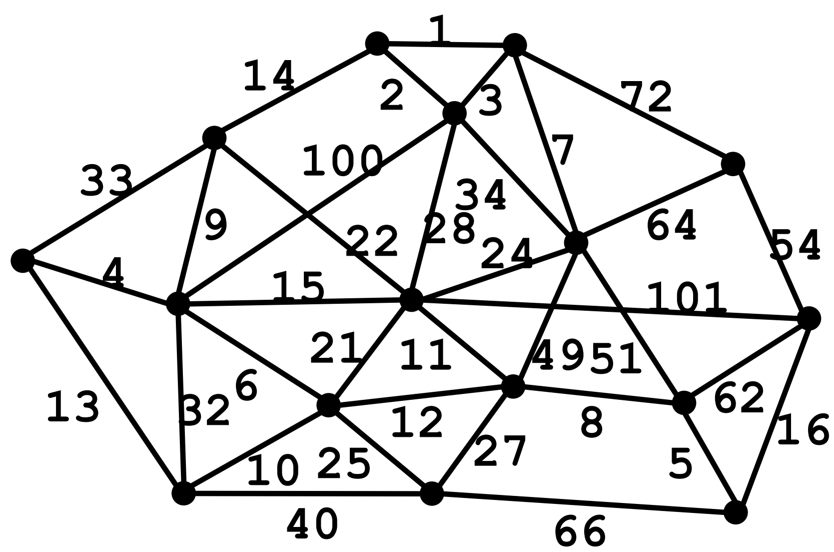
- Some graphs have exactly one minimum spanning tree. Others have several trees with the same cost, any of which is a minimum spanning tree

Minimum spanning trees

- Suppose edges are weighted (> 0), and we want a spanning tree of *minimum cost* (sum of edge weights)

- Useful in network routing & other applications

- For example, to stream a video



Greedy algorithm

A greedy algorithm: follow the heuristic of making a locally optimal choice at each stage, with the hope of finding a global optimum

Example. Make change using the fewest number of coins.
Make change for n cents, n < 100 (i.e. < $1)
Greedy: At each step, choose the largest possible coin

If n >= 50 choose a half dollar and reduce n by 50;
If n >= 25 choose a quarter and reduce n by 25;
As long as n >= 10, choose a dime and reduce n by 10;
If n >= 5, choose a nickel and reduce n by 5;
Choose n pennies.

Greedy algorithm

A greedy algorithm: follow the heuristic of making a locally optimal choice at each stage, with the hope of fining a global optimum. Doesn't always work

Example. Make change using the fewest number of coins.
Coins have these values: 7, 5, 1
Greedy: At each step, choose the largest possible coin

Consider making change for 10.
The greedy choice would choose: 7, 1, 1, 1.
But 5, 5 is only 2 coins.

Greedy algorithm

A greedy algorithm: follow the heuristic of making a locally optimal choice at each stage, with the hope of fining a global optimum. Doesn't always work

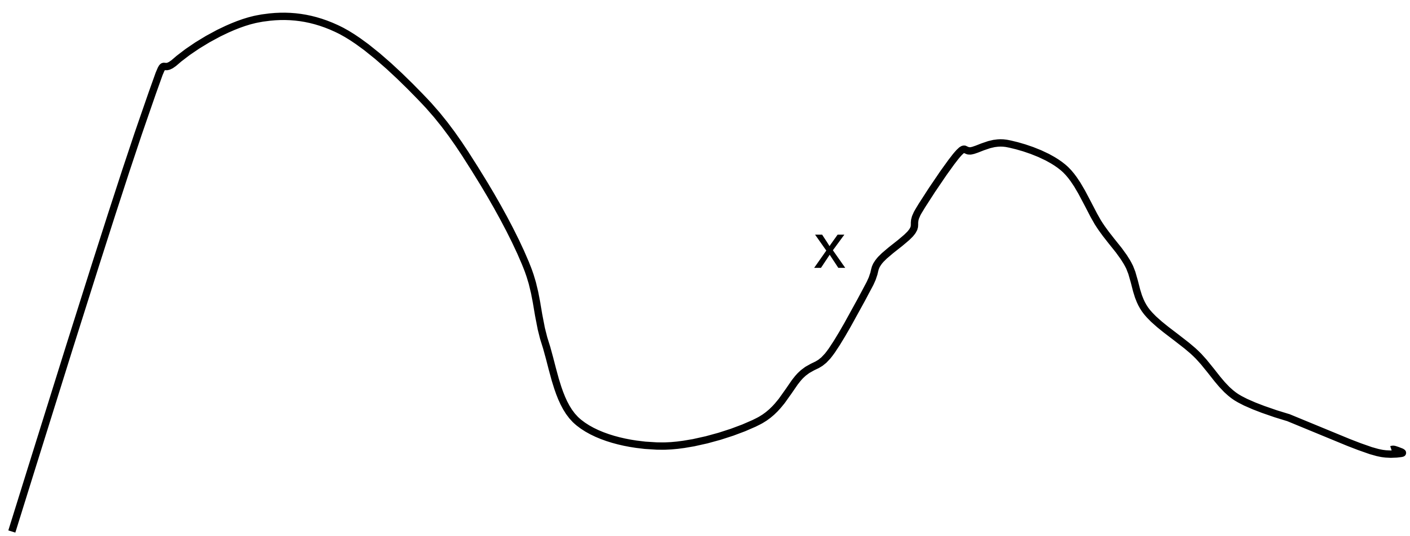
Example. Make change (if possible) using the fewest number of coins.
Coins have these values: 7, 5, 2
Greedy: At each step, choose the largest possible coin

Consider making change for 10.
The greedy choice would choose: 7, 2 –and can't proceed!
But 5, 5 works

Greediness doesn't work here

You're standing at point x, and your goal is to climb the highest mountain.

Two possible steps: down the hill or up the hill. The greedy step is to walk up hill. But that is a local optimum choice, not a global one. Greediness fails in this case.



Construct minimum spanning tree (greedy)

Maximal set of edges that contains no cycle

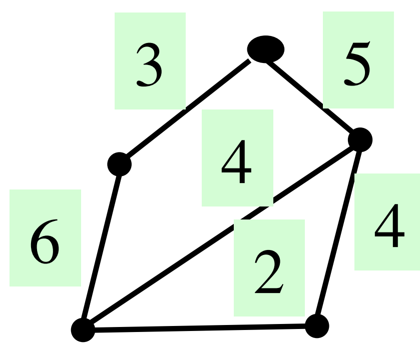
As long as there is a cycle:

Find a black max-weight edge –

if it is on a cycle, throw it out

otherwise keep it (make it red)

We mark a node red to indicate that we have looked at it and determined it can't be removed because removing it would unconnect the graph (the node is not on a cycle)



Construct minimum spanning tree (greedy)

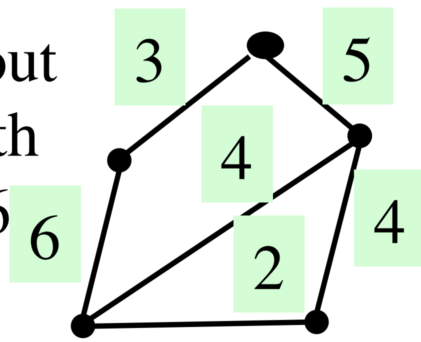
Maximal set of edges that contains no cycle

Nondeterministic algorithm

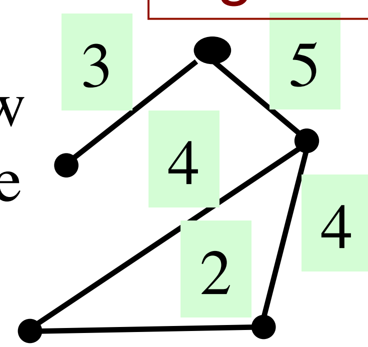
As long as there is a cycle:

Find a black max-weight edge – if it is on a cycle, throw it out otherwise keep it (make it red)

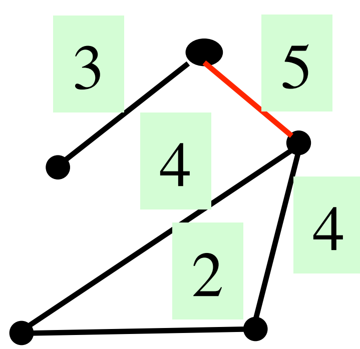
Throw out edge with weight 6

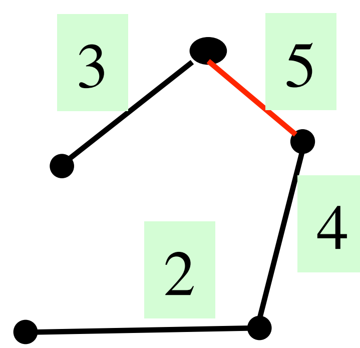


Can't throw out 5; make it red



Throw out one 4





No more cyles: done

Construct minimum spanning tree (greedy)

As long as there is a cycle:
  Find a black max-weight edge – if it is on a cycle, throw it out otherwise keep it (make it red)

Maximal set of edges that contains no cycle

Nobody uses this algorithm because, usually, there are far more edges than nodes. If graph with n nodes is complete, O(n*n) edges have to be deleted!

It's better to use this property of a spanning tree and add edges to the spanning tree. For a tree with n nodes, n-1 edges have to be added

Minimal set of edges that connect all vertices

Two greedy algorithms for constructing a minimum spanning tree

- Kruskal
- Prim

Both use this definition of a spanning tree and in a greedy fashion:

> Minimal set of edges that connect all vertices

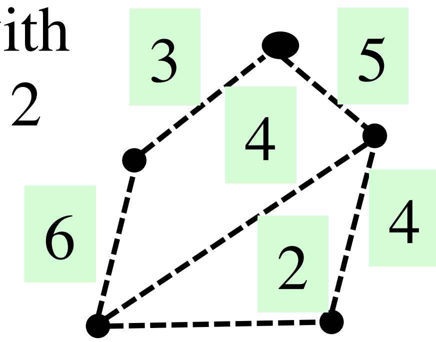
Both are nondeterministic, in that at a point they may choose one of several nodes with equal weight

Kruskal's algorithm: greedy

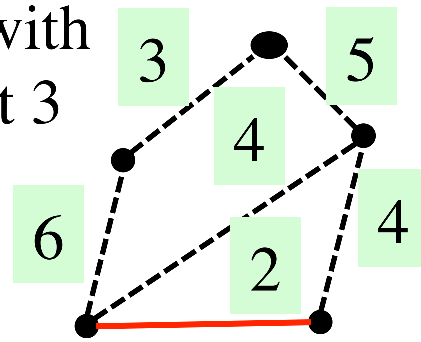
Minimal set of edges that connect all vertices

At each step, add an edge (that does not form a cycle) with minimum weight

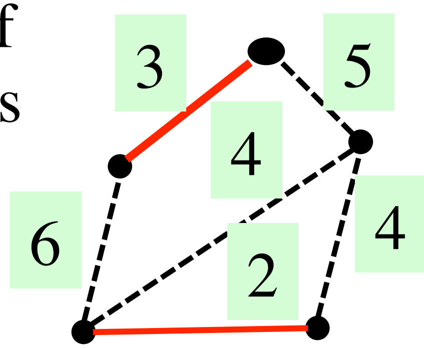
edge with weight 2



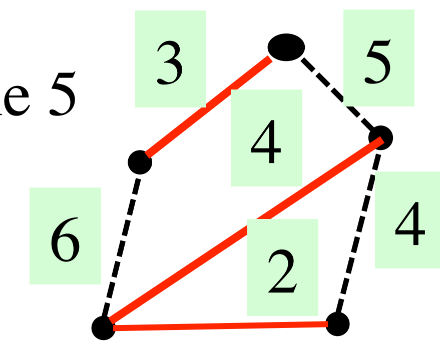
edge with weight 3

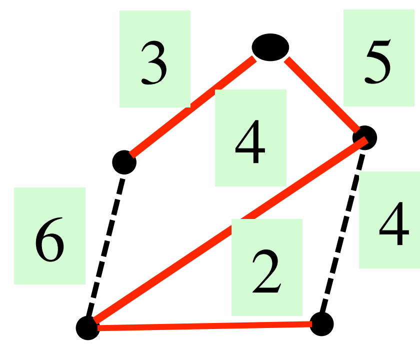


one of the 4's



the 5





Dashed edges: original graph

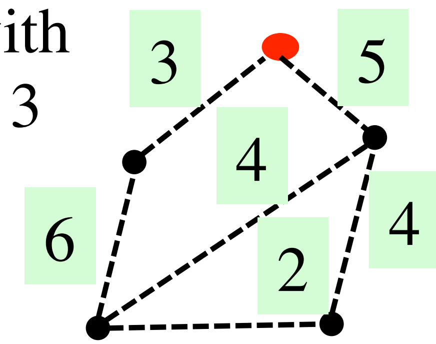
Red edges: the constructed spanning tree

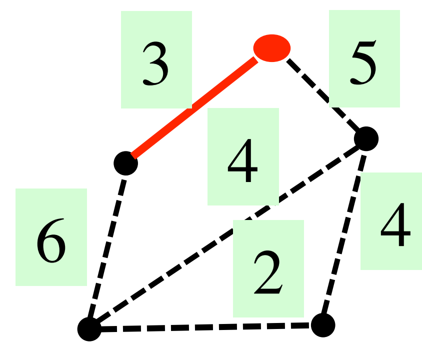
Prim's algorithm. greedy

Invariant: The added edges (and their nodes) are connected

Have start node.

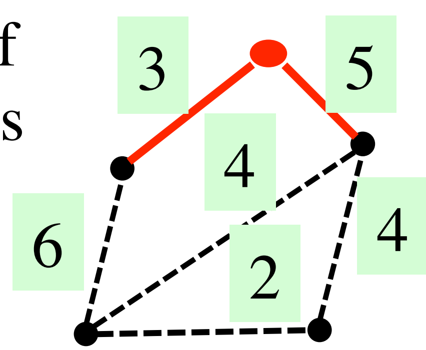
edge with weight 3



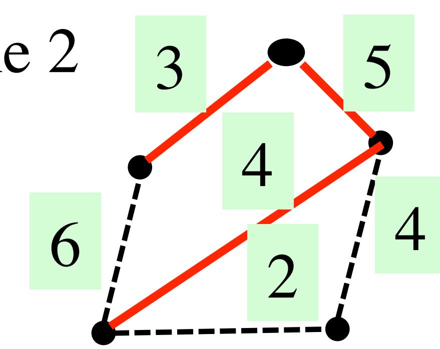


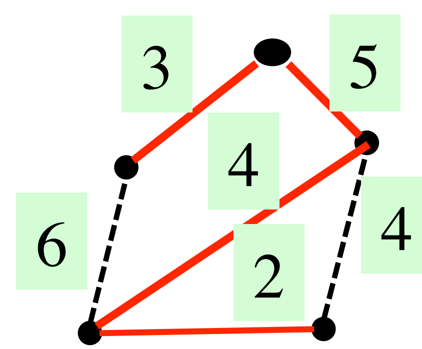
edge with weight 5

one of the 4's



the 2





Tree greedy spanning tree algorithms

1. Algorithm that uses this property of a spanning tree: Maximal set of edges that contains no cycle

2. Algorithms that use this property of a spanning tree: Minimal set of edges that connect all vertices
   (a) Kruskal (b) Prim

When edge weights are all distinct, or if there is exactly one minimum spanning tree, all 3 algorithms construct the same tree.

Prim's algorithm (n nodes, m edges)

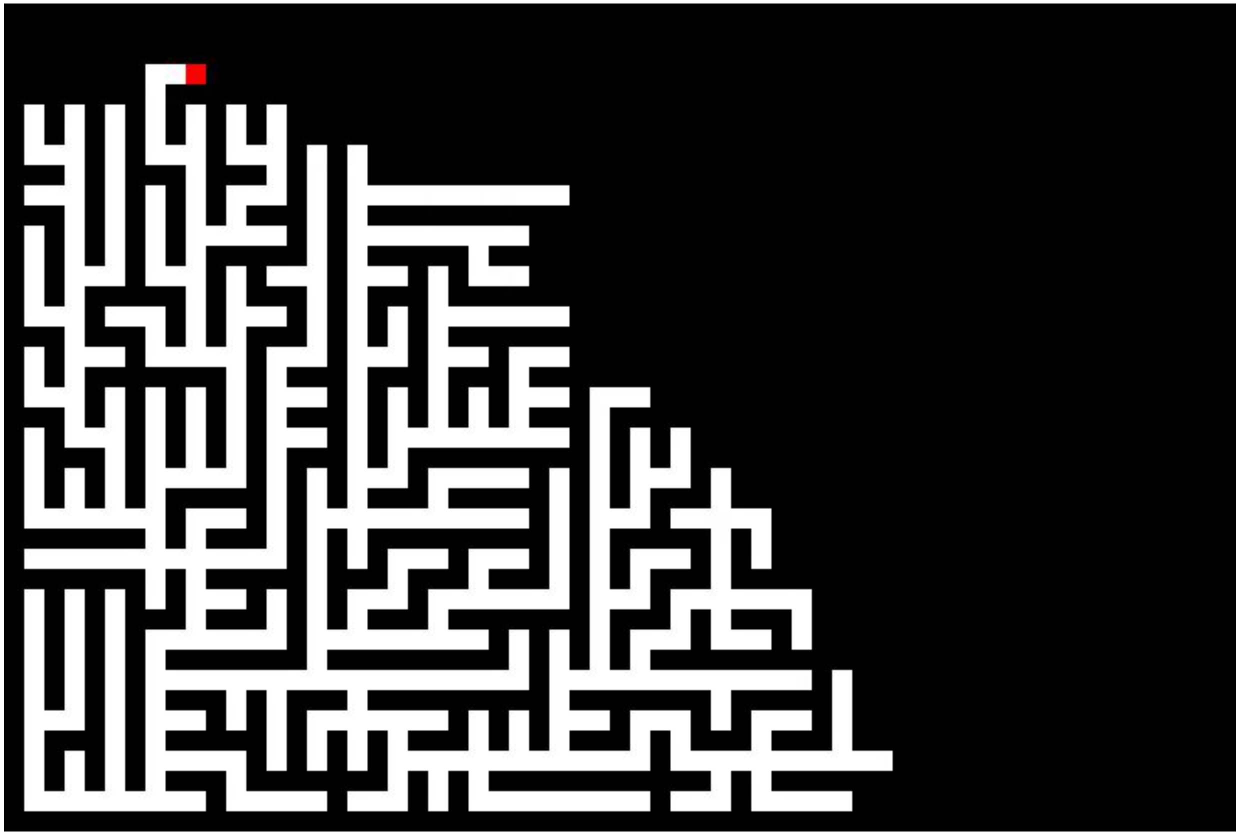

```
prim(s) {
  D[s]= 0; //start vertex
  D[i]= ∞ for all i ≠ s;
  while (a vertex is unmarked) {
    v= unmarked vertex
            with smallest D;
    mark v;
    for (each w adj to v)
      D[w]= min(D[w], c(v,w));
  }
}
```

- O(m + n log n) for adj list
  - Use a priority queue PQ
  - Regular PQ produces time O(n + m log m)
  - Can improve to O(m + n log n) using a fancier heap

- $O(n^2)$ for adj matrix
  - while-loop iterates n times
  - for-loop takes O(n) time

Application of MST

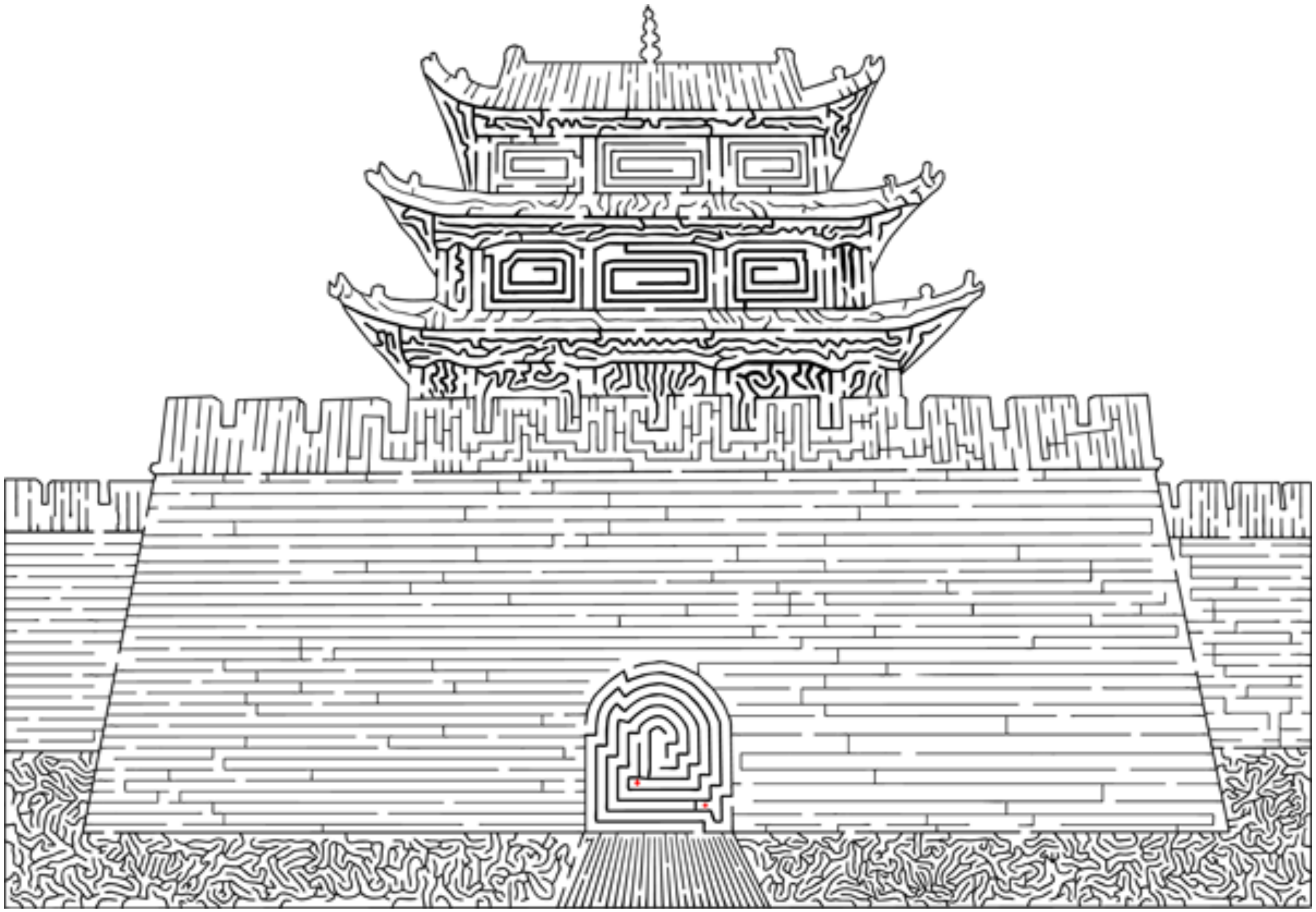
Maze generation using Prim's algorithm



The generation of a maze using Prim's algorithm on a randomly weighted grid graph that is 30x20 in size.

http://en.wikipedia.org/wiki/File:MAZE_30x20_Prim.ogv

More complicated maze generation



http://www.cgl.uwaterloo.ca/~csk/projects/mazes/

Greedy algorithms

- These are Greedy Algorithms
- Greedy Strategy: is an algorithm design technique
  - Like Divide & Conquer
- Greedy algorithms are used to solve optimization problems
  - Goal: find the *best* solution
- Works when the problem has the greedy-choice property:
  - A global optimum can be reached by making locally optimum choices

Example: Making change

Given an amount of money, find smallest number of coins to make that amount

Solution: Use Greedy Algorithm:

Use as many large coins as you can.

Produces optimum number of coins for US coin system

May fail for old UK system

Similar code structures


```
while (a vertex is unmarked) {
    v= best unmarked vertex
    mark v;
    for (each w adj to v)
        update D[w];
}
```

- Breadth-first-search (bfs)
  - best: next in queue
  - update: D[w] = D[v]+1
- Dijkstra's algorithm
  - best: next in priority queue
  - update: D[w] = min(D[w], D[v] +c(v,w))
- Prim's algorithm
  - best: next in priority queue
  - update: D[w] = min(D[w], c(v,w))

c(v,w) is the v→w edge weight

Traveling salesman problem

Given a list of cities and the distances between each pair, what is the shortest route that visits each city exactly once and returns to the origin city?

- The true TSP is very hard (called NP complete)… for this we want the ***perfect*** answer in all cases.
- Most TSP algorithms start with a spanning tree, then "evolve" it into a TSP solution. Wikipedia has a lot of information about packages you can download…