Overview references to sections in text

- Note: We’ve covered everything in JavaSummary.pptx!
- What is recursion? 7.1-7.39 slide 1-7
- Base case 7.1-7.10 slide 13
- How Java stack frames work 7.8-7.10 slide 28-32
A little about generics – used in A3

public class DLinkedList<E> { …}  // E is a type parameter

/** Values in d1 can be ANY objects — String, JFrame, etc. */
DLinkedList d1 = new DLinkedList();
…
String x = ((String) d1.getHead()).getValueOf();  // cast is needed

/** The values in d2 are only objects of class String */
DLinkedList<String> d2 = new DLinkedList<String>();
…
String s = d2.getHead().getValueOf();  // no cast is needed
What does generic mean?

*From Merriam-Webster online:*

**generic adjective**

*a*: relating or applied to or descriptive of all members of a genus, species, class, or group: common to or characteristic of a whole group or class: typifying or subsuming: not specific or individual

*generic* applies to that which characterizes every individual in a category or group and may suggest further that what is designated may be thought of as a clear and certain classificatory criterion
Sum the digits in a non-negative integer

/** return sum of digits in n.  
 * Precondition:  n >= 0 */

public static int sum(int n) {
    if (n < 10) return n;

    // { n has at least two digits }
    // return first digit + sum of rest
    return sum(n/10) + n%10;
}

E.g. sum(7) = 7

E.g. sum(8703) = sum(870) + 3;
Two issues with recursion

1. Why does it work? How does it execute?

```java
/** return sum of digits in n. * Precondition: n >= 0 */
public static int sum(int n) {
    if (n < 10) return n;
    // { n has at least two digits }
    // return first digit + sum of rest
    return sum(n/10) + n%10 + ;
}
```

2. How do we understand a given recursive method, or how do we write/develop a recursive method?
Stacks and Queues

Stack: list with (at least) two basic ops:
* Push an element onto its top
* Pop (remove) top element

Last-In-First-Out (LIFO)

Like a stack of trays in a cafeteria

Queue: list with (at least) two basic ops:
* Append an element
* Remove first element

First-In-First-Out (FIFO)

Americans wait in a line, the Brits wait in a queue!
A “frame” contains information about a method call:

At runtime, Java maintains a stack that contains frames for all method calls that are being executed but have not completed.

Method call: push a frame for call on stack, assign argument values to parameters, execute method body. Use the frame for the call to reference local variables, parameters.

End of method call: pop its frame from the stack; if it is a function, leave the return value on top of stack.
Example: Sum the digits in a non-negative integer

```java
public static int sum(int n) {
    if (n < 10) return n;
    return sum(n/10) + n%10;
}

public static void main(…) {
    int r= sum(824);
    System.out.println(r);
}
```

Frame for method in the system that calls method main
Example: Sum the digits in a non-negative integer

```java
public static int sum(int n) {
    if (n < 10) return n;
    return sum(n/10) + n%10;
}

public static void main(…) {
    int r = sum(824);
    System.out.println(r);
}
```

Frame for method in the system that calls method main: main is then called
Example: Sum the digits in a non-negative integer

```java
public static int sum(int n) {
    if (n < 10) return n;
    return sum(n/10) + n%10 ;
}

public static void main(...) {
    int r= sum(824);
    System.out.println(r);
}
```

Method main calls sum:

```
n 824
return info
```
```
main
r ___ args ___
return info
```
```
system
? 
return info
```
Example: Sum the digits in a non-negative integer

```java
public static int sum(int n) {
    if (n < 10) return n;
    return sum(n/10) + n%10;
}

public static void main(…) {
    int r = sum(824);
    System.out.println(r);
}
```

n >= 10, sum calls sum:
public static int sum(int n) {
    if (n < 10) return n;
    return sum(n/10) + n%10;
}

public static void main(…) {
    int r = sum(824);
    System.out.println(r);
}

n >= 10. sum calls sum:
Example: Sum the digits in a non-negative integer

```java
public static int sum(int n) {
    if (n < 10) return n;
    return sum(n/10) + n%10;
}

public static void main(…) {
    int r = sum(824);
    System.out.println(r);
}
```

n < 10, sum stops: frame is popped and n is put on stack:
Example: Sum the digits in a non-negative integer

```java
public static int sum(int n) {
    if (n < 10) return n;
    return sum(n/10) + n%10;
}

public static void main(…) {
    int r = sum(824);
    System.out.println(r);
}
```

Using return value 8, stack computes $8 + 2 = 10$, pops frame from stack, puts return value 10 on stack
Example: Sum the digits in a non-negative integer

```java
public static int sum(int n) {
    if (n < 10) return n;
    return sum(n/10) + n%10;
}

public static void main(…) {
    int r= sum(824);
    System.out.println(r);
}
```

Using return value 10, stack computes
10 + 4 = 14, pops frame from stack,
puts return value 14 on stack
Example: Sum the digits in a non-negative integer

```java
public static int sum(int n) {
    if (n < 10) return n;
    return sum(n/10) + n%10;
}

public static void main(...) {
    int r = sum(824);
    System.out.println(r);
}
```

Using return value 14, main stores 14 in r and removes 14 from stack.
Summary of method call execution

Memorize this!

1. A frame for a call contains parameters, local variables, and other information needed to properly execute a method call.

2. To execute a method call: push a frame for the call on the stack, assign arg values to pars, and execute method body.

   When executing method body, look in frame for call for parameters and local variables.

   When method body finishes, pop frame from stack and (for a function) push the return value on the stack.

3. For function call: When control given back to call, it pops the return value and uses it as the value of the function call.
In a call m(),
when is local variable d created and when is it destroyed?
Which version of procedure m do you like better? Why?
Recursion is used extensively in math

Math definition of n factorial

\[ \begin{align*}
0! &= 1 \\
n! &= n \times (n-1)! \quad \text{for } n > 0
\end{align*} \]

E.g. \(3! = 3 \times 2 \times 1 = 6\)

Math definition of \(b^c\) for \(c \geq 0\)

\[ \begin{align*}
b^0 &= 1 \\
b^c &= b \times b^{c-1} \quad \text{for } c > 0
\end{align*} \]

Easy to make math definition into a Java function!

```java
public static int fact(int n) {
    if (n == 0) return 1;
    return n * fact(n-1);
}
```

Lots of things defined recursively: expression, grammars, trees, ….
We will see such things later
Two views of recursive methods

- How are calls on recursive methods executed? We saw that. Use this only to gain understanding / assurance that recursion works.

- How do we understand a recursive method — know that it satisfies its specification? How do we write a recursive method? This requires a totally different approach. Thinking about how the method gets executed will confuse you completely! We now introduce this approach.
Understanding a recursive method

Step 1. Have a precise spec!

Step 2. Check that the method works in the base case(s): Cases where the parameter is small enough that the result can be computed simply and without recursive calls.

If \( n < 10 \), then \( n \) consists of a single digit. Looking at the spec, we see that that digit is the required sum.

```java
/** = sum of digits of \( n \).
 * Precondition: \( n >= 0 \) */
public static int sum(int n) {
    if (n < 10) return n;
    // \( n \) has at least two digits
    return sum(n/10) + n%10;
}
```
Understanding a recursive method

Step 1. Have a precise spec!

Step 2. Check that the method works in the base case(s).

Step 3. Look at the recursive case(s). In your mind, replace each recursive call by what it does according to the method spec and verify that the correct result is then obtained.

// e.g. n = 843

```java
/** = sum of digits of n. 
 * Precondition:  n >= 0 */

public static int sum(int n) {
    if (n < 10)
        return n;
    // n has at least two digits
    return sum(n/10) + n%10;
}
```

return sum(n/10) + n%10;

return (sum of digits of n/10) + n%10;
Understanding a recursive method

Step 1. Have a precise spec!

Step 2. Check that the method works in the base case(s).

Step 3. Look at the recursive case(s). In your mind, replace each recursive call by what it does acc. to the spec and verify correctness.

Step 4. (No infinite recursion) Make sure that the args of recursive calls are in some sense smaller than the pars of the method.

```java
/** = sum of digits of n.  *
 * Precondition: n >= 0 */

public static int sum(int n) {
    if (n < 10) return n;
    // n has at least two digits
    return sum(n/10) + n%10;
}
```

\[ n/10 < n \]
Understanding a recursive method

Step 1. Have a precise spec!

Step 2. Check that the method works in the base case(s).

Step 3. Look at the recursive case(s). In your mind, replace each recursive call by what it does according to the spec and verify correctness.

Step 4. (No infinite recursion) Make sure that the args of recursive calls are in some sense smaller than the pars of the method.

Important! Can’t do step 3 without it

Once you get the hang of it, this is what makes recursion easy!
This way of thinking is based on math induction, which we will see later in the course.
Writing a recursive method

Step 1. Have a precise spec!

Step 2. Write the base case(s): Cases in which no recursive calls are needed. Generally, for “small” values of the parameters.

Step 3. Look at all other cases. See how to define these cases in terms of smaller problems of the same kind. Then implement those definitions, using recursive calls for those smaller problems of the same kind. Done suitably, point 4 is automatically satisfied.

Step 4. (No infinite recursion) Make sure that the args of recursive calls are in some sense smaller than the pars of the method.
Examples of writing recursive functions

For the rest of the class, we demo writing recursive functions using the approach outlined below. The java file we develop will be placed on the course webpage some time after the lecture.

Step 1. Have a precise spec!

Step 2. Write the base case(s).

Step 3. Look at all other cases. See how to define these cases in terms of smaller problems of the same kind. Then implement those definitions, using recursive calls for those smaller problems of the same kind.
The Fibonacci Function

Mathematical definition:
\[ \text{fib}(0) = 0 \]
\[ \text{fib}(1) = 1 \]
\[ \text{fib}(n) = \text{fib}(n - 1) + \text{fib}(n - 2), \quad n \geq 2 \]

Fibonacci sequence: 0, 1, 1, 2, 3, 5, 8, 13, ...

/** = fibonacci(n). Pre: n >= 0 */
static int fib(int n) {
    if (n <= 1) return n;
    // { 1 < n }
    return fib(n-2) + fib(n-1);
}
Example: Is a string a palindrome?

```java
/** = "s is a palindrome" */
public static boolean isPal(String s) {
    if (s.length() <= 1)
        return true;

    // { s has at least 2 chars }
    int n = s.length() - 1;
    return s.charAt(0) == s.charAt(n) && isPal(s.substring(1, n));
}

isPal("racecar") returns true
isPal("pumpkin") returns false
```
Example: Count the e’s in a string

```java
/** = number of times c occurs in s */
public static int countEm(char c, String s) {
    if (s.length() == 0) return 0;
    // { s has at least 1 character }
    if (s.charAt(0) != c)
        return countEm(c, s.substring(1));
    // { first character of s is c }
    return 1 + countEm(c, s.substring(1));
}
```

- countEm('e', "it is easy to see that this has many e’s") = 4
- countEm('e', "Mississippi") = 0
Computing $a^n$ for $n \geq 0$

Power computation:
- $a^0 = 1$
- If $n \neq 0$, $a^n = a \cdot a^{n-1}$
- If $n \neq 0$ and even, $a^n = (a \cdot a)^{n/2}$

Java note: For ints $x$ and $y$, $x/y$ is the integer part of the quotient

Judicious use of the third property gives a logarithmic algorithm, as we will see

Example: $3^8 = (3 \cdot 3) \cdot (3 \cdot 3) \cdot (3 \cdot 3) \cdot (3 \cdot 3) = (3 \cdot 3)^4$
Computing $a^n$ for $n \geq 0$

Power computation:
- $a^0 = 1$
- If $n \neq 0$, $a^n = a \times a^{n-1}$
- If $n \neq 0$ and even, $a^n = (a \times a)^{n/2}$

```c
/** = a**n. Precondition: n >= 0 */
static int power(int a, int n) {
    if (n == 0) return 1;
    if (n%2 == 0) return power(a*a, n/2);
    return a * power(a, n-1);
}
```
Recursion is a convenient and powerful way to define functions.

Problems that seem insurmountable can often be solved in a “divide-and-conquer” fashion:

- Reduce a big problem to smaller problems of the same kind, solve the smaller problems
- Recombine the solutions to smaller problems to form solution for big problem
Extra material: memoization

Execution of \( \text{fib}(4) \) is inefficient. E.g. in the tree to right, you see 3 calls of \( \text{fib}(1) \).

To speed it up, save values of \( \text{fib}(i) \) in a table as they are calculated. For each \( i \), \( \text{fib}(i) \) called only once. The table is called a cache.

```c
/** = fibonacci(n), for n >= 0 */
static int fib(int n) {
    if (n <= 1) return n;
    // { 1 < n }
    return fib(n-2) + fib(n-1);
}
```
Memoization: an optimization technique used to speed up execution by having function calls avoid repeating the calculation of results for previously processed inputs.

- The first time the function is called, save result
- The next time, look the result up

- Assumes a “side effect free” function: The function just computes the result, it doesn’t change things
- If the function depends on anything that changes, must “empty” the saved results list
Adding memoization to our solution

Before memoization:

```java
static int fib(int n) {
    int v = n <= 1 ? n : fib(n-1) + fib(n-2);
    return v;
}
```

```
/** For 0 <= k < cached.size(), cached[k] = fib(k) */
static ArrayList<Integer> cached = new ArrayList<Integer>();
```
/** For 0 <= k < cached.size(), cached[k] = fib(k) */

static ArrayList<Integer> cached = new ArrayList<Integer>();

static int fib(int n) {
    if (n < cached.size()) return cached.get(n);

    int v = n <= 1 ? n : fib(n-1) + fib(n-2);

    if (n == cached.size())
        cached.add(v);

    return v;
}
Memoization uses a static field

Recursive functions should NOT use static fields. It just doesn’t work. Don’t try to write recursive functions that way. The one case that it works is suitably written memoization.

```java
/** For 0 <= k < cached.size(), cached[k] = fib(k) */
static ArrayList<Integer> cached = new ArrayList<Integer>();

static int fib(int n) {
    if (n < cached.size()) return cached.get(n);
    int v = n <= 1 ? n : fib(n-1) + fib(n-2);
    if (n == cached.size()) cached.add(v);
    return v;
}
```