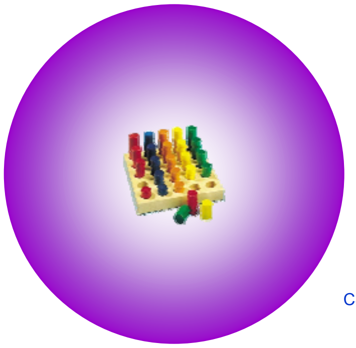

Solving Recurrences

Lecture 13
CS2110 – Summer 2008

Announcements

- Prelim tomorrow!
  - Closed book
  - Try to arrive a little early
  - Exam will begin promptly at 10:00, end at 11:15
  - Topics include everything covered up to and including last Friday's class
- Assignment 2 is graded
  - Regrade requests due Friday (??)
  - Solutions available on CMS
- Assignment 3 solutions?

Analysis of Merge-Sort

```
public static Comparable[] mergeSort(Comparable[] A, int low, int high) {
   if (low < high) { //at least 2 elements?          cost = c
      int mid = (low + high)/2;                      cost = d
      Comparable[] A1 = mergeSort(A, low, mid);      cost = T(n/2) + e
      Comparable[] A2 = mergeSort(A, mid+1, high);   cost = T(n/2) + f
      return merge(A1,A2);                           cost = gn + h
   }                                                 cost = i
   ....
```

Recurrence:

$T(n) = c + d + e + f + 2T(n/2) + gn + h$ ← recurrence
$T(1) = i$ ← base case

How do we solve this recurrence?

Analysis of Merge-Sort

Recurrence:

$T(n) = c + d + e + f + 2T(n/2) + gn + h$
$T(1) = i$

First, simplify by dropping lower-order terms

Simplified recurrence:

$T(n) = 2T(n/2) + gn$
$T(1) = i$

How do we find the solution?

Solving Recurrences

- Unfortunately, solving recurrences is like solving differential equations
  - No general technique works for all recurrences
- Luckily, can get by with a few common patterns
- You will learn some more techniques in CS 280

Analysis of Merge-Sort

- Recurrence for MergeSort
  - $T(n) = 2T(n/2) + cn$
  - $T(2) = 2c$
  - Solution is $T(n) = O(n \log n)$

- Proof: strong induction on n
- Show that
  $T(2) \le 2c$
  $T(n) \le 2T(n/2) + cn$
  imply
  $T(n) \le cn \log n$
- Basis
  $T(2) \le 2c = c\,2 \log 2$
- Induction step
  $T(n) \le 2T(n/2) + cn$
  $\le 2(cn/2 \log n/2) + cn$ (IH)
  $= cn (\log n – 1) + cn$
  $= cn \log n$

Solving Recurrences

- Recurrences are important when designing divide & conquer algorithms

- Solution techniques:
  - Can sometimes change variables to get a simpler recurrence
  - Make a guess, then prove the guess correct by induction
  - Build a recursion tree and use it to determine solution
  - Can use the *Master Method*
    - A "cookbook" scheme that handles many common recurrences

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Master Method

- To solve recurrences of the form

$$T(n) = aT(n/b) + f(n),$$

with constants $a \geq 0$, $b > 1$, compare $f(n)$ with $n^{\log_b a}$

  - if f(n) grows more rapidly,
    - Solution is $T(n) = O(f(n))$
  - if $n^{\log_b a}$ grows more rapidly
    - Solution is $T(n) = O(n^{\log_b a})$
  - if both grow at same rate
    - Solution is $T(n) = O(f(n) \log n)$

- Not an exact statement of the theorem – f(n) must be "well-behaved"

Recurrence Examples

- $T(n) = T(n-1) + 1$ → $T(n) = O(n)$ Linear Search
- $T(n) = T(n-1) + n$ → $T(n) = O(n^2)$ QuickSort worst-case
- $T(n) = T(n/2) + 1$ → $T(n) = O(\log n)$ Binary Search
- $T(n) = T(n/2) + n$ → $T(n) = O(n)$
- $T(n) = 2\,T(n/2) + n$ → $T(n) = O(n \log n)$ MergeSort
- $T(n) = 2\,T(n-1)$ → $T(n) = O(2^n)$
- $T(n) = 4\,T(n/2) + n$ → $T(n) = O(n^2)$

	10	50	100	300	1000
$5n$	50	250	500	1500	5000
$n\log n$	33	282	665	2469	9966
n^2	100	2500	10,000	90,000	1,000,000
n^3	1000	125,000	1,000,000	27 million	1 billion
2^n	1024	a 16-digit number	a 31-digit number	a 91-digit number	a 302-digit number
$n!$	3.6 million	a 65-digit number	a 161-digit number	a 623-digit number	unimaginably large
n^n	10 billion	an 85-digit number	a 201-digit number	a 744-digit number	unimaginably large

- protons in the known universe ~ 126 digits
- µsec since the big bang ~ 24 digits

- Source: D. Harel, *Algorithmics*

How long would it take @ 1 instruction / µsec ?

	10	20	50	100	300
n^2	1/10,000 sec	1/2500 sec	1/400 sec	1/100 sec	9/100 sec
n^5	1/10 sec	3.2 sec	5.2 min	2.8 hr	28.1 days
2^n	1/1000 sec	1 sec	35.7 yr	400 trillion centuries	a 75-digit number of centuries
n^n	2.8 hr	3.3 trillion years	a 70-digit number of centuries	a 185-digit number of centuries	a 728-digit number of centuries

- The big bang was 15 billion years ago ($5 \cdot 10^{17}$ secs)

- Source: D. Harel, *Algorithmics*

The Fibonacci Function

- Mathematical definition:

$fib(0) = 0$
$fib(1) = 1$
$fib(n) = fib(n-1) + fib(n-2), \quad n \geq 2$

- Fibonacci sequence: 0, 1, 1, 2, 3, 5, 8, 13, …

```
static int fib(int n) {
   if (n == 0) return 0;
   else if (n == 1) return 1;
   else return fib(n-1) + fib(n-2);
}
```

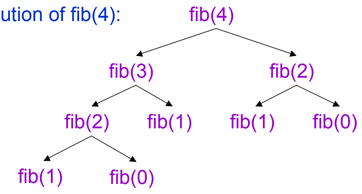
Fibonacci (Leonardo Pisano) 1170–1240?

Statue in Pisa, Italy
Giovanni Paganucci
1863

Recursive Execution

```
static int fib(int n) {
   if (n == 0) return 0;
   else if (n == 1) return 1;
   else return fib(n-1) + fib(n-2);
}
```

Execution of fib(4):



The Fibonacci Recurrence

```
static int fib(int n) {
   if (n == 0) return 0;
   else if (n == 1) return 1;
   else return fib(n-1) + fib(n-2);
}
```

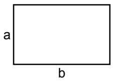
$T(0) = c$

$T(1) = c$

$T(n) = T(n – 1) + T(n – 2) + c$

- Solution is exponential in n
- But not quite $O(2^n)$...

The Golden Ratio



ratio of sum of sides (a+b) to longer side (b)

=

ratio of longer side (b) to shorter side (a)

$\varphi = (a+b)/b = b/a$

$\varphi^2 = \varphi + 1$

$\varphi = \frac{1+\sqrt{5}}{2}$

$= 1.618...$

Fibonacci Recurrence is $O(\varphi^n)$

- want to show $T(n) \le c\varphi^n$
- have $\varphi^2 = \varphi + 1$
- multiplying by $c\varphi^n$, $c\varphi^{n+2} = c\varphi^{n+1} + c\varphi^n$
- Basis:
  - $T(0) = c = c\varphi^0$
  - $T(1) = c \le c\varphi^1$
- Induction step:
  - $T(n+2) = T(n+1) + T(n) \le c\varphi^{n+1} + c\varphi^n = c\varphi^{n+2}$

Can We Do Better?

```
if (n <= 1) return n;
int parent = 0;
int current = 1;
for (int i = 2; i ≤ n; i++) {
   int next = current + parent;
   parent = current;
   current = next;
}
return (current);
```

- Number of times loop is executed? Less than n
- Number of basic steps per loop? Constant
- Complexity of iterative algorithm = O(n)
- Much, much, much, much, much, better than $O(\varphi^n)$!

...But We Can Do Even Better!

- Let f_n denote the nth Fibonacci number
  - $f_0 = 0$
  - $f_1 = 1$
  - $f_{n+2} = f_{n+1} + f_n,\ n \ge 0$
- Note that $\begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} f_n \\ f_{n+1} \end{pmatrix} = \begin{pmatrix} f_{n+1} \\ f_{n+2} \end{pmatrix}$, thus $\begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}^n\begin{pmatrix} f_0 \\ f_1 \end{pmatrix} = \begin{pmatrix} f_n \\ f_{n+1} \end{pmatrix}$
- Can compute the nth power of a matrix by repeated squaring in O(log n) time
- Gives complexity O(log n)
- Just a little cleverness got us from exponential to logarithmic!

Matrix Multiplication in Less Than $O(n^3)$ (Strassen's Algorithm)

- Idea: naive 2 x 2 matrix multiplication takes 8 scalar multiplications, but we can do it in 7:

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} e & f \\ g & h \end{pmatrix} = \begin{pmatrix} s_1 + s_2 - s_4 + s_6 & s_4 + s_5 \\ s_6 + s_7 & s_2 - s_3 + s_5 - s_7 \end{pmatrix}$$

where

$s_1 = (b - d)(g + h)$
$s_2 = (a + d)(e + h)$
$s_3 = (a - c)(e + f)$
$s_4 = h(a + b)$
$s_5 = a(f - h)$
$s_6 = d(g - e)$
$s_7 = e(c + d)$

Now Apply This Recursively – Divide and Conquer!

- Break $2^{n+1} \times 2^{n+1}$ matrices up into 4 $2^n \times 2^n$ submatrices
- Multiply them the same way

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix}\begin{pmatrix} E & F \\ G & H \end{pmatrix} = \begin{pmatrix} S_1 + S_2 - S_4 + S_6 & S_4 + S_5 \\ S_6 + S_7 & S_2 - S_3 + S_5 - S_7 \end{pmatrix}$$

where

$S_1 = (B - D)(G + H)$
$S_2 = (A + D)(E + H)$
$S_3 = (A - C)(E + F)$
$S_4 = H(A + B)$
$S_5 = A(F - H)$
$S_6 = D(G - E)$
$S_7 = E(C + D)$

Now Apply This Recursively – Divide and Conquer!

- Gives recurrence $M(n) = 7\,M(n/2) + cn^2$ for the number of multiplications
- Solution is $M(n) = O(n^{\log 7}) = O(n^{2.81\ldots})$
- Number of additions is $O(n^2)$, bound separately

Is That the Best You Can Do?

- How about 3 x 3 for a base case?
  - best known is 23 multiplications
  - not good enough to beat Strassen
- In 1978, Victor Pan discovered how to multiply 70 x 70 matrices with 143640 multiplications, giving $O(n^{2.795\ldots})$
- Best bound to date (obtained by entirely different methods) is $O(n^{2.376\ldots})$ (Coppersmith & Winograd 1987)
- Best known lower bound is still $\Omega(n^2)$

Moral: Complexity Matters!

- But you are acquiring the best tools to deal with it!