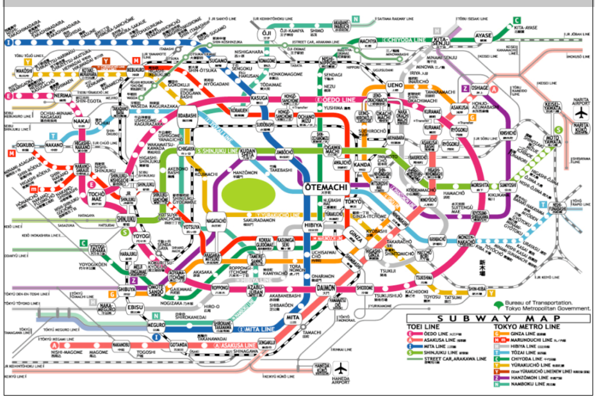
More Graphs

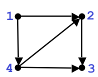
Lecture 21
CS211 – Summer 2007



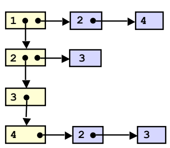
Announcements

- Office hour changes for this week
  – David's OH from 3-4pm instead of 2-3
  – Additional consulting hours tomorrow 12-2pm
- Tomorrow's class

Representations of Graphs



Adjacency List



Adjacency Matrix

	1	2	3	4
1	0	1	0	1
2	0	0	1	0
3	0	0	0	0
4	0	1	1	0

Adjacency Matrix or Adjacency List?

n = number of vertices
m = number of edges
d(u) = outdegree of u

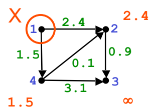
- Adjacency Matrix
  – Uses space $O(n^2)$
  – Can iterate over all edges in time $O(n^2)$
  – Can answer "Is there an edge from u to v?" in O(1) time
  – Better for dense graphs (lots of edges)
- Adjacency List
  – Uses space O(m+n)
  – Can iterate over all edges in time O(m+n)
  – Can answer "Is there an edge from u to v?" in O(d(u)) time
  – Better for sparse graphs (fewer edges)

Shortest Paths in Graphs

- Finding the shortest (min-cost) path in a graph is a problem that occurs often
  – Find the shortest route between Ithaca and West Lafayette, IN
  – Result depends on our notion of cost
    - Least mileage
    - Least time
    - Cheapest
    - Least boring
  – All of these "costs" can be represented as edge weights
- How do we find a shortest path?

Dijkstra's Algorithm

```
dijkstra(s) {
   D[s] = 0; D[t] = c(s,t), t ≠ s;
   mark s;
   while (some vertices are unmarked) {
      v = unmarked node with smallest D;
      mark v;
      for (each w adjacent to v) {
         D[w] = min(D[w], D[v] + c(v,w));
      }
   }
}
```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2.4
1.5
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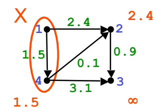
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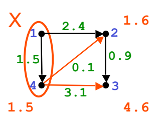
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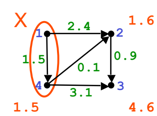
4.6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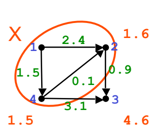
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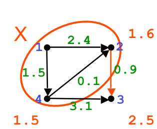
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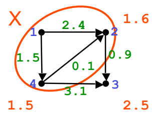
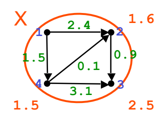




Proof of Correctness

The following are invariants of the loop:

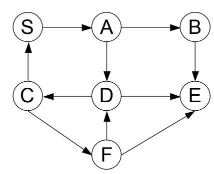
- For $u \in X$, $D(u) = d(s,u)$
- For $u \in X$ and $v \notin X$, $d(s,u) \leq d(s,v)$
- For all u, D(u) is the length of the shortest path from s to u such that all nodes on the path (except possibly u) are in X

Implementation:

- Use a priority queue for the nodes not yet taken – priority is D(u)

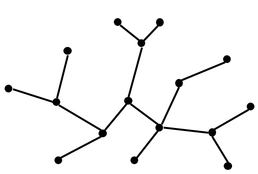
Shortest Paths for Unweighted Graphs – A Special Case

- Use breadth-first search
- Time is $O(n + m)$ in adj list representation, $O(n^2)$ in adj matrix representation



Undirected Trees

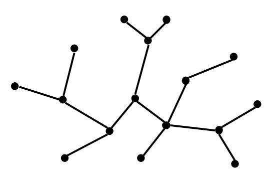
- An undirected graph is a *tree* if there is exactly one simple path between any pair of vertices



Facts About Trees

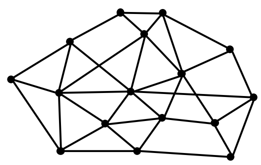
- $|E| = |V| - 1$
- connected
- no cycles

In fact, any two of these properties imply the third, and imply that the graph is a tree



Spanning Trees

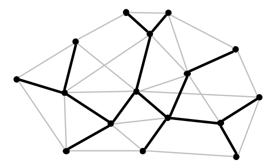
A *spanning tree* of a connected undirected graph (V,E) is a subgraph (V,E') that is a tree



Spanning Trees

A *spanning tree* of a connected undirected graph (V,E) is a subgraph (V,E') that is a tree

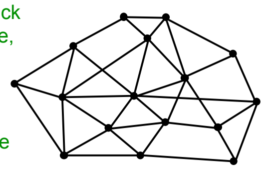
- Same set of vertices V
- $E' \subseteq E$
- (V,E') is a tree



Finding a Spanning Tree

A subtractive method

- Start with the whole graph – it is connected
- If there is a cycle, pick an edge on the cycle, throw it out – the graph is still connected (why?)
- Repeat until no more cycles



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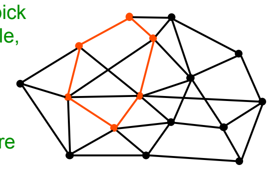

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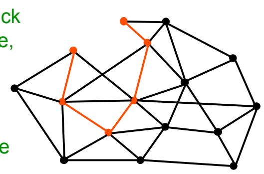
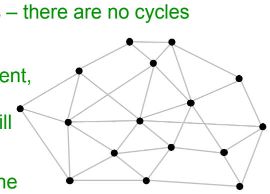


Finding a Spanning Tree

An additive method

- Start with no edges – there are no cycles
- If more than one connected component, insert an edge between them – still no cycles (why?)
- Repeat until only one component



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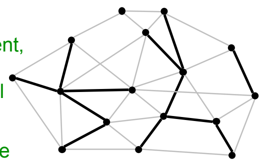

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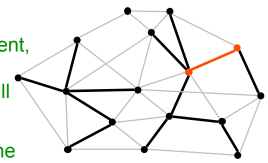

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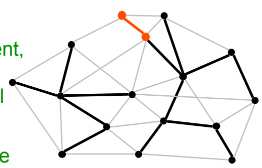

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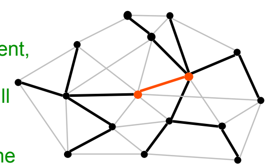

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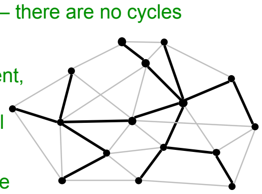
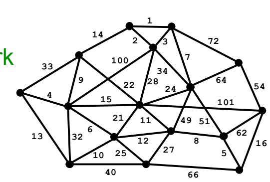


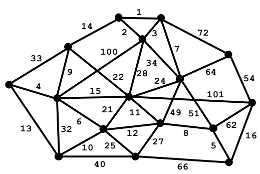
Minimum Spanning Trees

- Suppose edges are weighted, and we want a spanning tree of *minimum cost* (sum of edge weights)
- Useful in network routing & other applications



3 Greedy Algorithms

A. Find a max weight edge – if it is on a cycle, throw it out, otherwise keep it



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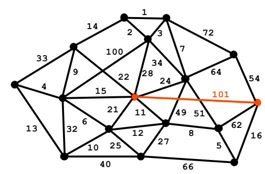

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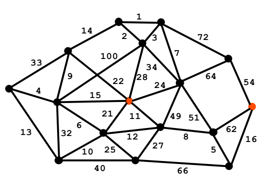

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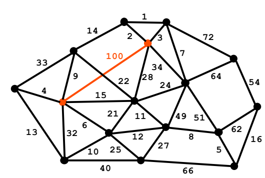

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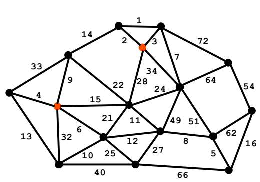

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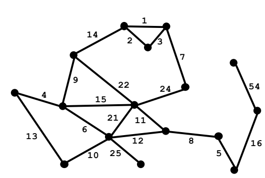

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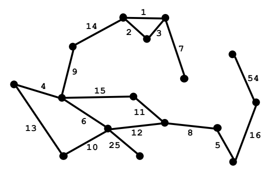

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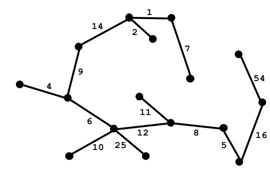
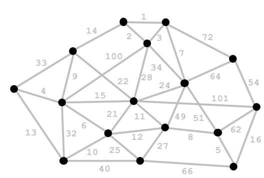


3 Greedy Algorithms

B. Find a min weight edge – if it forms a cycle with edges already taken, throw it out, otherwise keep it

Kruskal's algorithm



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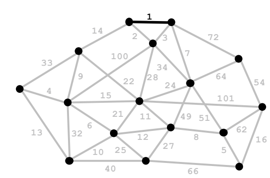

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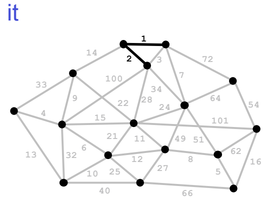

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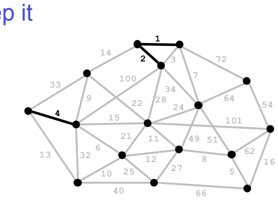

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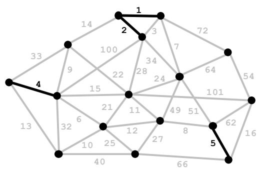

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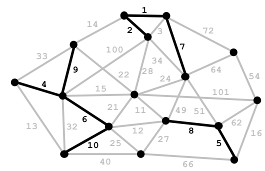

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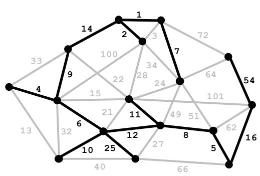
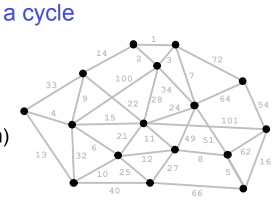


3 Greedy Algorithms

C. Start with any vertex, add min weight edge extending that connected component that does not form a cycle

Prim's algorithm
(reminiscent of Dijkstra's algorithm)



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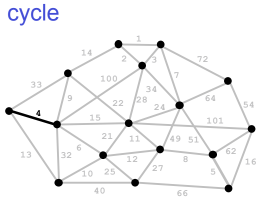

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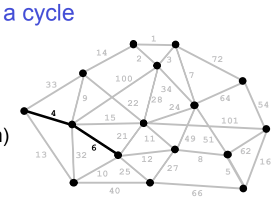

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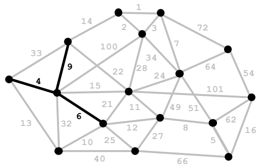

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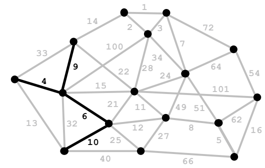

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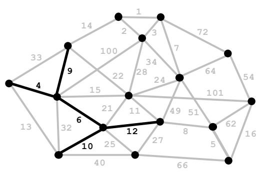

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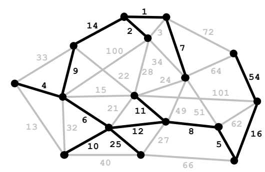
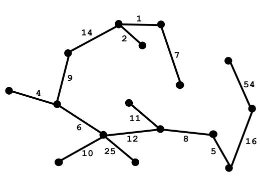


3 Greedy Algorithms

All 3 greedy algorithms give the same minimum spanning tree (assuming distinct edge weights)



Prim's Algorithm

```
prim(s) {
   D[s] = 0; mark s; //start vertex
   while (some vertices are unmarked) {
      v = unmarked vertex with smallest D;
      mark v;
      for (each w adj to v) {
         D[w] = min(D[w], c(v,w));
      }
   }
}
```

- $O(n^2)$ for adj matrix
  - While-loop is executed n times
  - For-loop takes O(n) time
- O(m + n log n) for adj list
  - Use a PQ
  - Regular PQ produces time O(n + m log m)
  - Can improve to O(m + n log n) using a fancier heap

Greedy Algorithms

- These are examples of Greedy Algorithms
- The Greedy Strategy is an algorithm design technique
  - Like Divide & Conquer
- Greedy algorithms are used to solve optimization problems
  - The goal is to find the *best* solution
- Works when the problem has the greedy-choice property
  - A global optimum can be reached by making locally optimum choices
- Example: the Change Making Problem: Given an amount of money, find the smallest number of coins to make that amount
- Solution: Use a Greedy Algorithm
  - Give as many large coins as you can
- This greedy strategy produces the optimum number of coins for the US coin system
- Different money system ⇒ greedy strategy may fail
  - Example: old UK system

Similar Code Structures

```
while (some vertices are
       unmarked) {
 v = best of unmarked
      vertices;
 mark v;
 for (each w adj to v)
       update w;
}
```

- bfs
  - best: next in queue
  - update: D[w] = D[v]+1
- dijkstra
  - best: next in PQ
  - update: D[w] = min D[w], D[v]+c(v,w)
- prim
  - best: next in PQ
  - update: D[w] = min D[w], c(v,w)