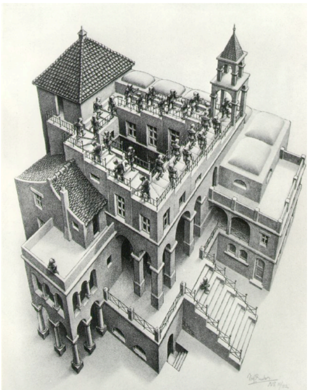


Priority Queues and Heaps

Lecture 15
CS211 Summer 2007

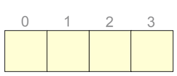
Announcements

- Assignment 3 due tonight, 11:59
- Prelim grades on CMS

Hash table example

h("january")=105 h("april")=160 h("july")=123
h("february")=623 h("may)=147 h("august)=137
h("march")=139 h("june")=108 h("september")=109

m=4
n=0
λ=n/m=0
λ_0=0.7

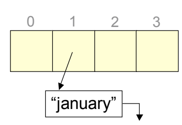


Hash table example

h("january")=105 h("april")=160 h("july")=123
h("february")=623 h("may)=147 h("august)=137
h("march")=139 h("june")=108 h("september")=109

Insert("january") [105 mod m = 1]

m=4
n=1
λ=n/m=0.25
λ_0=0.7

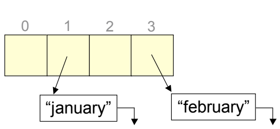


Hash table example

h("january")=105 h("april")=160 h("july")=123
h("february")=623 h("may)=147 h("august)=137
h("march")=139 h("june")=108 h("september")=109

Insert("february") [623 mod m = 3]

m=4
n=2
λ=n/m=0.5
λ_0=0.7

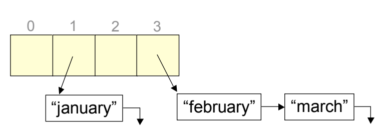


Hash table example

h("january")=105 h("april")=160 h("july")=123
h("february")=623 h("may)=147 h("august)=137
h("march")=139 h("june")=108 h("september")=109

Insert("march") [139 mod m = 3]

m=4
n=3
λ=n/m=0.75
λ_0=0.7



Hash table example

h(“january”)=105 h(“april”)=160 h(“july”)=123

h(“february”)=623 h(“may)=147 h(“august)=137

h(“march”)=139 h(“june”)=108 h(“september”)=109

[105 mod m = 1]

[623 mod m = 7]

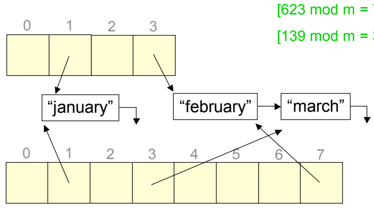
[139 mod m = 3]

m=8

n=3

λ=n/m=0.375

λ_0=0.7



Hash table example

h(“january”)=105 h(“april”)=160 h(“july”)=123

h(“february”)=623 h(“may)=147 h(“august)=137

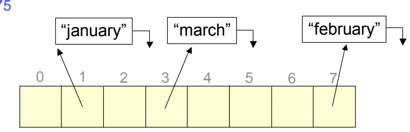
h(“march”)=139 h(“june”)=108 h(“september”)=109

m=8

n=3

λ=n/m=0.375

λ_0=0.7



Hash table example

h(“january”)=105 h(“april”)=160 h(“july”)=123

h(“february”)=623 h(“may)=147 h(“august)=137

h(“march”)=139 h(“june”)=108 h(“september”)=109

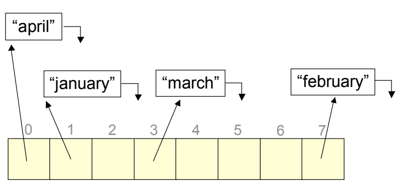
Insert(“april”) [160 mod m = 0]

m=8

n=4

λ=n/m=0.5

λ_0=0.7



Hash table example

h(“january”)=105 h(“april”)=160 h(“july”)=123

h(“february”)=623 h(“may)=147 h(“august)=137

h(“march”)=139 h(“june”)=108 h(“september”)=109

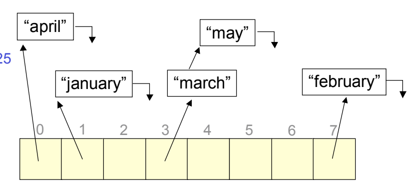
Insert(“may”) [147 mod m = 3]

m=8

n=5

λ=n/m=0.625

λ_0=0.7



Hash table example

h(“january”)=105 h(“april”)=160 h(“july”)=123

h(“february”)=623 h(“may)=147 h(“august)=137

h(“march”)=139 h(“june”)=108 h(“september”)=109

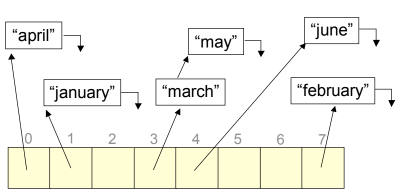
Insert(“june”) [108 mod m = 4]

m=8

n=6

λ=n/m=0.75

λ_0=0.7



Hash table example

h(“january”)=105 h(“april”)=160 h(“july”)=123

h(“february”)=623 h(“may)=147 h(“august)=137

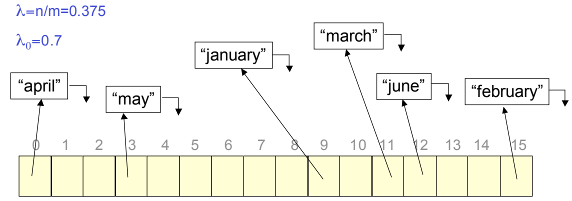
h(“march”)=139 h(“june”)=108 h(“september”)=109

m=16

n=6

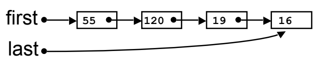
λ=n/m=0.375

λ_0=0.7



Stacks and Queues as Lists

- Stack (LIFO) implemented as list
  - `put()`, `get()` from front of list
- Queue (FIFO) implemented as list
  - `put()` on back of list, `get()` from front of list



Priority Queue

- *lesser* elements (as determined by `compareTo()`) have *higher* priority
- `get()` returns the element with the highest priority = least in the `compareTo()` ordering
- break ties arbitrarily

Examples

- Scheduling jobs to run on a computer
  - default priority = arrival time
  - priority can be changed by operator
- Scheduling events to be processed by an event handler
  - priority = time of occurrence
- Airline check-in
  - first class, business class, coach
  - FIFO within each class

Priority Queues

```
java.util.PriorityQueue<E>

boolean add(E e) {...} //insert an element (put)
void clear() {...} //remove all elements
E peek() {...} //return min element without removing
               //(null if empty)
E poll() {...} //remove min element (get)
               //(null if empty)
int size() {...}
```

Priority Queues as Lists

- Maintain as unordered list
  - `add()` puts new element at front – O(1)
  - `poll()` must search the list – O(n)
- Maintain as ordered list
  - `add()` must search the list – O(n)
  - `poll()` gets element at front – O(1)
- In either case, $O(n^2)$ to process n elements

Can we do better?

Important Special Case

- Fixed number of priority levels 0,...,p – 1
- FIFO within each level
- Example: airline check-in, O/S scheduling

- `add()` – insert in appropriate queue – O(1)
- `poll()` – must find a nonempty queue – O(p)

Heaps

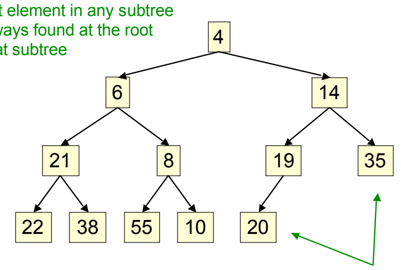
- A *heap* is a concrete data structure that can be used to implement priority queues
- Gives better complexity than either ordered or unordered list implementation:
  - `add()`, `poll()`: O(log n)
  - `size()`: O(1)
- O(n log n) to process n elements
- Do not confuse with *heap memory*, where the Java virtual machine allocates space for objects – different usage of the word *heap*

Heaps

- Binary tree with data at each node
- Satisfies the *Heap Order Invariant*:

> The least (highest priority) element of any subtree is found at the root of that subtree

Least element in any subtree is always found at the root of that subtree



But it is possible to have smaller elements deeper in the tree!

Examples of Heaps

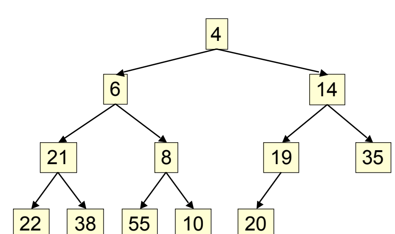
- Ages of people in family tree
  - parent is always older than children, but you can have an uncle who is younger than you
- Salaries of employees of a company
  - bosses generally make more than subordinates, but a VP in one subdivision may make less than a Project Supervisor in a different subdivision

Balanced Heaps

Two restrictions:

1. Any node of depth < d – 1 has exactly 2 children, where d is the height of the tree
   - implies that any path from a root to a leaf is either of length d or d – 1
   - Also implies that the tree has at least 2^d nodes
2. All maximal paths of length d are to the left of those of length d – 1

A Balanced Heap



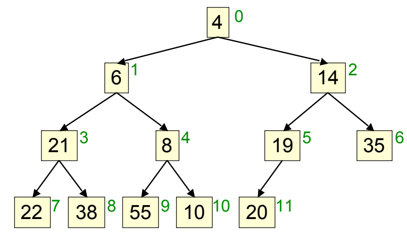
d = 3

Store in an Array or Vector

- Elements of the heap are stored in array in order, going across each level from left to right, top to bottom
- The children of the node at array index n are found at 2n + 1 and 2n + 2
- The parent of node n is found at (n – 1)/2

Heaps can be stored in an array

0	1	2	3	4	5	6	7	8	9	10	11
4	6	14	21	8	19	35	22	38	55	10	20

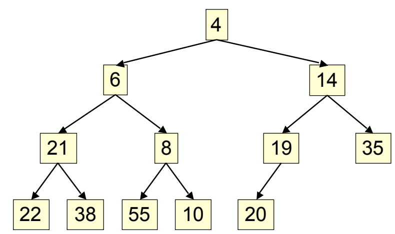


children of node n are found at 2n + 1 and 2n + 2

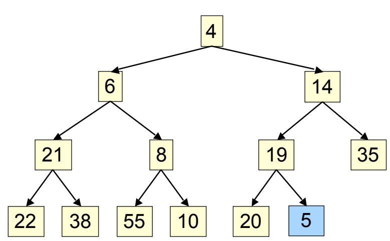
`add()`

- Put the new element at the end of the array
- If this violates heap order because it is smaller than its parent, swap it with its parent
- Continue swapping it up until it finds its rightful place
- The heap invariant is maintained!

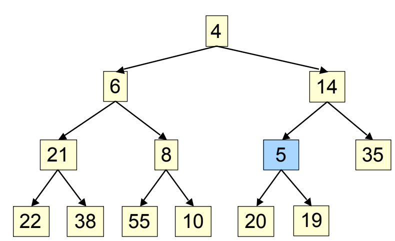
`add()`



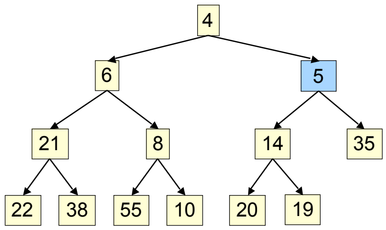
`add()`



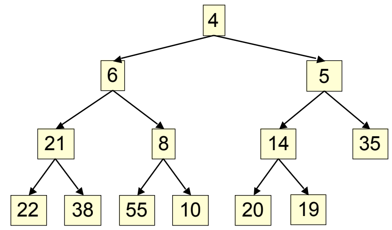
`add()`



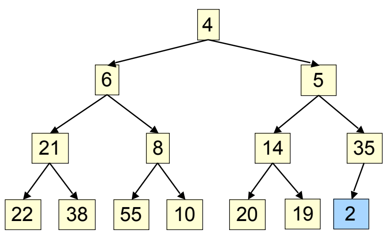
`add()`



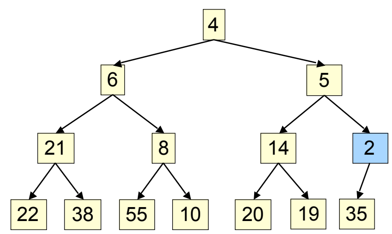
`add()`



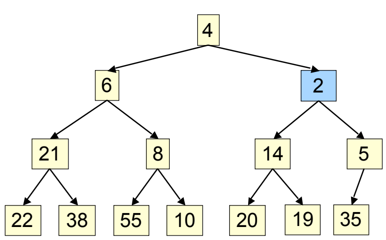
`add()`



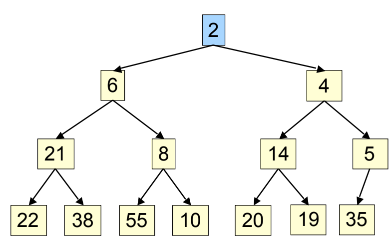
`add()`



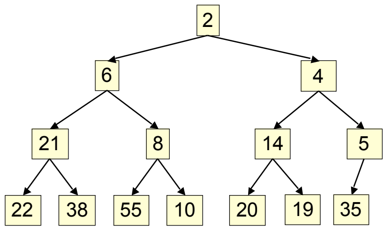
`add()`



`add()`



add()



add()

- Time is O(log n), since the tree is balanced
  - size of tree is exponential as a function of depth
  - depth of tree is logarithmic as a function of size

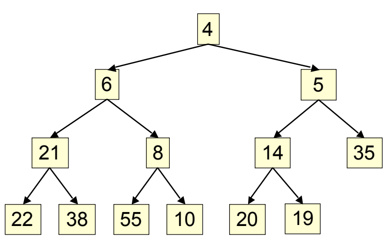
add()

```
class PriorityQueue<E> extends java.util.Vector<E> {

  public void add(E obj) {
    super.add(obj); //add new element to end of array
    rotateUp(size() - 1);
  }

  private void rotateUp(int index) {
    if (index == 0) return;
    int parent = (index - 1)/2;
    if (elementAt(parent).compareTo(elementAt(index)) <= 0)
      return;
    swap(index, parent);
    rotateUp(parent);
  }
```

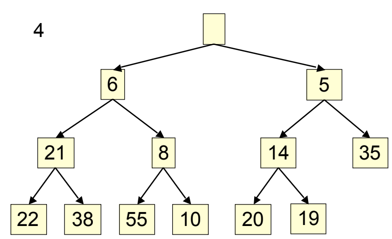
poll()

- Remove the least element – it is at the root
- This leaves a hole at the root – fill it in with the last element of the array
- If this violates heap order because the root element is too big, swap it down with the smaller of its children
- Continue swapping it down until it finds its rightful place
- The heap invariant is maintained!

poll()

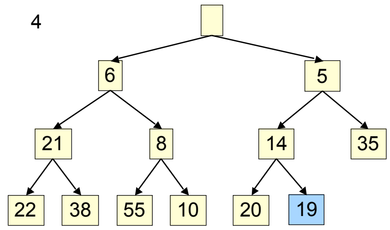


poll()



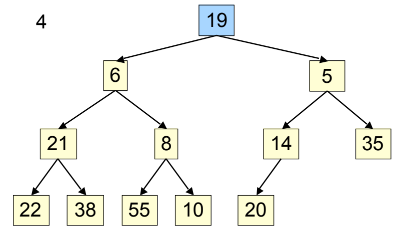
poll()

4



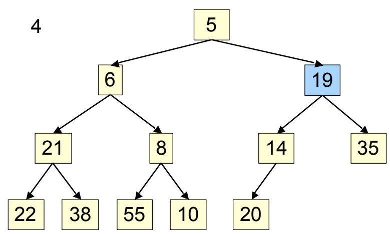
poll()

4



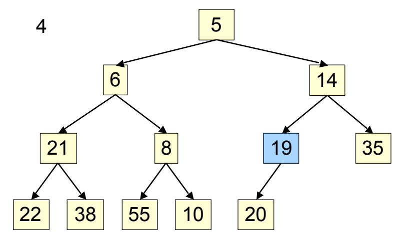
poll()

4



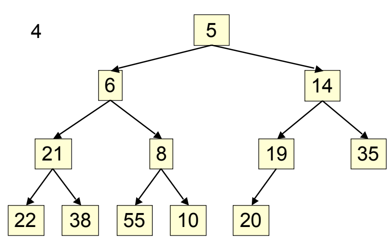
poll()

4



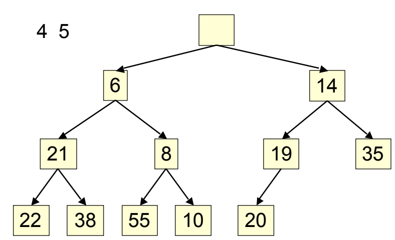
poll()

4

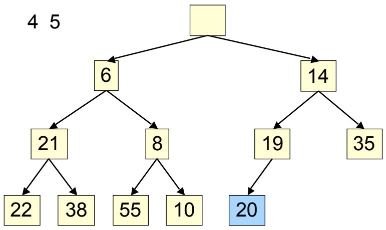


poll()

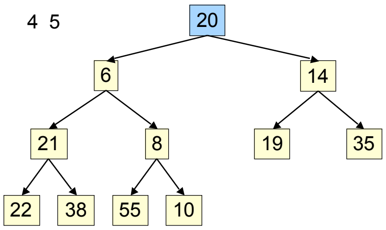
4 5



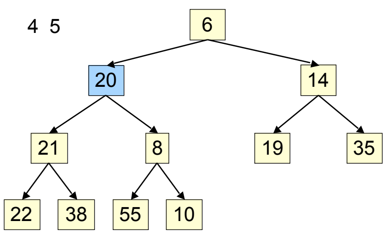
poll()



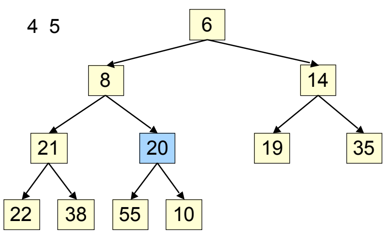
poll()



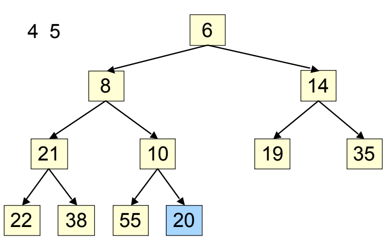
poll()



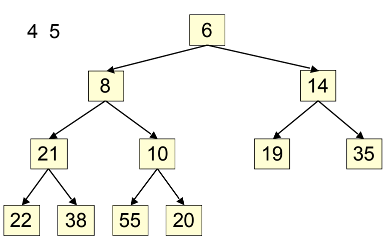
poll()



poll()



poll()



`poll()`

- Time is O(log n), since the tree is balanced

`poll()`

```
public E poll() {
  if (size() == 0) return null;
  E temp = elementAt(0);
  setElementAt(0, elementAt(size() - 1));
  setSize(size() - 1);
  rotateDown(0);
  return temp;
}
private void rotateDown(int index) {
  int child = 2*(index + 1); //right child
  if (child >= size()
    || elementAt(child - 1).compareTo(elementAt(child)) < 0)
      child -= 1;
  if (child >= size()) return;
  if (elementAt(index).compareTo(elementAt(child)) <= 0)
    return;
  swap(index, child);
  rotateDown(child);
}
```

HeapSort

Given a `Comparable[]` array of length n,

1. Put all n elements into a heap – O(n log n)
2. Repeatedly get the min – O(n log n)

```
public static void heapSort(Comparable[] a) {
   PriorityQueue<Comparable> pq
      = new PriorityQueue<Comparable>();
   for (Comparable x : a) { pq.add(x); }
   for (int i = 0; i < a.length; i++) { a[i] = pq.poll(); }
}
```

PQ Application: Simulation

- Example: Probabilistic model of bank-customer arrival times and transaction times, how many tellers are needed?
  - Assume we have a way to generate random inter-arrival times
  - Assume we have a way to generate transaction times
  - Can simulate the bank to get some idea of how long customers must wait

Time-Driven Simulation

- Check at each *tick* to see if any event occurs

Event-Driven Simulation

- Advance clock to next event, skipping intervening *ticks*
- This uses a PQ!