Induction

Lecture 6
CS211 – Summer 2007

Overview

- Recursion
  - A programming strategy that solves a problem by reducing it to simpler or smaller instance(s) of the same problem
- Induction
  - A mathematical strategy for proving statements about natural numbers 0,1,2,... (or more generally, about inductively defined objects)
- Induction and recursion are very closely related

Defining Functions

- It is often useful to describe a function in different ways
  - Let S : int → int be the function where S(n) is the sum of the integers from 0 to n. For example,
    S(0) = 0 S(3) = 0+1+2+3 = 6
  - Definition: iterative form
    - $S(n) = 0+1+ \ldots + n = \sum_{i=0}^{n} i$
  - Recursive definition
    - S(0) = 0
    - S(n) = n + S(n-1) for n > 0
  - Another characterization: closed form
    - S(n) = n(n+1)/2

Sum of Squares

- A more complex example
  - Let SQ : int → int be the function that gives the sum of the squares of integers from 0 to n:
    $SQ(0) = 0 \quad SQ(3) = 0^2 + 1^2 + 2^2 + 3^2 = 14$
- Definition (iterative form): $SQ(n) = 0^2 + 1^2 + \ldots + n^2$
- Is there an equivalent closed-form expression?

Closed-Form Expression for SQ(n)

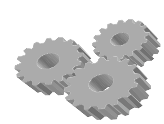
- Sum of integers between 0 through n was n(n+1)/2 which is a *quadratic* in n
- Inspired guess: perhaps sum of squares of integers between 0 through n is a *cubic* in n

- Conjecture: $SQ(n) = an^3+bn^2+cn+d$ where a, b, c, d are unknown coefficients
- How can we find the values of the four unknowns?
  - Idea: Use any 4 values of n to generate 4 linear equations, and then solve

Finding Coefficients

$$SQ(n) = 0^2+1^2+\ldots+n^2 = an^3+bn^2+cn+d$$

- Use n = 0, 1, 2, 3



$SQ(0) = 0 = a\cdot0 + b\cdot0 + c\cdot0 + d$
$SQ(1) = 1 = a\cdot1 + b\cdot1 + c\cdot1 + d$
$SQ(2) = 5 = a\cdot8 + b\cdot4 + c\cdot2 + d$
$SQ(3) = 14 = a\cdot27 + b\cdot9 + c\cdot3 + d$

Solve these 4 equations to get
a = 1/3 b = 1/2 c = 1/6 d = 0

Is the Formula Correct?

- This suggests

$SQ(n) = 0^2 + 1^2 + \ldots + n^2$
$= n^3/3 + n^2/2 + n/6$
$= n(n+1)(2n+1)/6$

- Question: Is this closed-form solution true for all n?
  - Remember, we only used n = 0,1,2,3 to determine these coefficients
  - We do not know that the closed-form expression is valid for other values of n

One Approach

- Try a few other values of n to see if they work.
  - Try n = 5:   SQ(n) = 0+1+4+9+16+25 = 55
  - Closed-form expression: 5·6·11/6 = 55
  - Works!
- Try some more values…
- We can never prove validity of the closed-form solution for all values of n this way, since there are an infinite number of values of n

A Recursive Definition

- To solve this problem, let's express SQ(n) in a different way:
  $SQ(n) = \boxed{0^2 + 1^2 + \ldots + (n-1)^2} + n^2$
  - The part in the box is just SQ(n-1)
- This leads to the following recursive definition
  $SQ(0) = 0$ ← Base Case
  $SQ(n) = SQ(n-1) + n^2, \quad n > 0$ ← Recursive Case
- Thus,
  $SQ(4) = SQ(3) + 4^2$
  $= SQ(2) + 3^2 + 4^2$
  $= SQ(1) + 2^2 + 3^2 + 4^2$
  $= SQ(0) + 1^2 + 2^2 + 3^2 + 4^2$
  $= 0 + 1^2 + 2^2 + 3^2 + 4^2$

Are These Two Functions Equal?

- SQ_r (r = recursive)

$SQ_r(0) = 0$
$SQ_r(n) = SQ_r(n-1) + n^2, \quad n > 0$

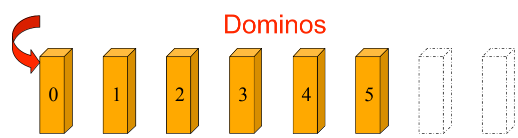
- SQ_c (c = closed-form)

$SQ_c(n) = n(n+1)(2n+1)/6$

Induction over Integers

- To prove that some property P(n) holds for all integers n ≥ 0,
  - Basis: Show that P(0) is true
  - Induction Step: Assuming that P(k) is true for an *unspecified* integer k, show that P(k+1) is true
- Conclusion: Because we could have picked any k, we conclude that P(n) holds for all integers n ≥ 0

Dominos



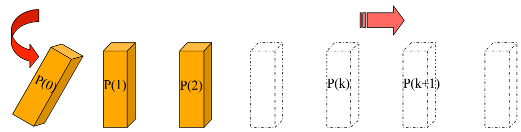
- Assume equally spaced dominos, and assume that spacing between dominos is less than domino length
- How would you argue that all dominos would fall?
- Dumb argument:
  - Domino 0 falls because we push it over
  - Domino 0 hits domino 1, therefore domino 1 falls
  - Domino 1 hits domino 2, therefore domino 2 falls
  - Domino 2 hits domino 3, therefore domino 3 falls
  - ...
- Is there a more compact argument we can make?

Better Argument

- Argument:
  - Domino 0 falls because we push it over (Base Case or Basis)
  - Assume that domino k falls over (Induction Hypothesis)
  - Because domino k's length is larger than inter-domino spacing, it will knock over domino k+1 (Inductive Step)
  - Because we could have picked any domino to be the k^{th} one, we conclude that all dominoes will fall over (Conclusion)
- This is an inductive argument
- This version is called *weak induction*
  - There is also *strong induction* (later)
- Not only is this argument more compact, it works for an arbitrary number of dominoes!

$SQ_r(n) = SQ_c(n)$ for all n?

- Define P(n) as $SQ_r(n)= SQ_c(n)$



- Prove P(0)
- Assume P(k) for unspecified k, and then prove P(k+1) under this assumption

Proof (by Induction)

$SQ_r(0) = 0$
$SQ_r(n) = SQ_r(n-1) + n^2, \quad n > 0$

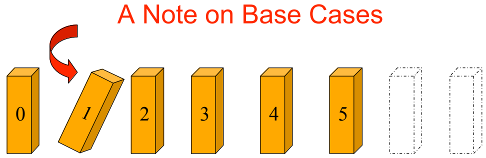
$SQ_c(n) = n(n+1)(2n+1)/6$

Let P(n) be the proposition $SQ_r(n) = SQ_c(n)$. Prove P(n) $\forall n \geq 0$.

Another Example

Prove that $0+1+...+n = n(n+1)/2 \quad \forall\ n \geq 0$.

A Note on Base Cases



- Sometimes we are interested in showing some proposition is true for integers ≥ b
- Intuition: we knock over domino b, and dominoes in front get knocked over; not interested in 0,1,…,(b–1)
- In general, the base case in induction does not have to be 0
- If base case is some integer b
  - Induction proves the proposition for n = b, b+1, b+2, …
  - Does not say anything about n = 0,1,…,b–1

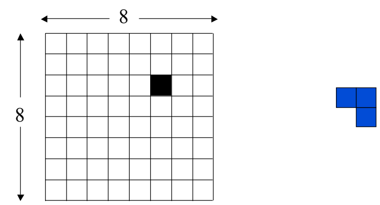
Weak Induction: Nonzero Base Case

Claim: You can make any amount of postage above 8¢ with some combination of 3¢ and 5¢ stamps

What are the “Dominos”?

- In some problems, it can be tricky to determine how to set up the induction
- This is particularly true for geometric problems that can be attacked using induction

A Tiling Problem



- A chessboard has one square cut out of it
- Can the remaining board be tiled using tiles of the shape shown in the picture (rotation allowed)?
- Not obvious that we can use induction!

Proof Outline

Consider boards of size 2^n x 2^n for n = 1,2,…

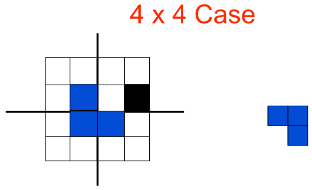
- Basis: Show that tiling is possible for 2 x 2 board
- Induction Hypothesis: Assume the 2^k x 2^k board can be tiled
- Inductive Step: Using I.H. show that the 2^{k+1} x 2^{k+1} board can be tiled
- Conclusion: Any 2^n x 2^n board can be tiled, n = 1,2,…
  - Our chessboard (8 x 8) is a special case of this argument
  - We will have proven the 8 x 8 special case by solving a more general problem!

Basis

2 x 2 board

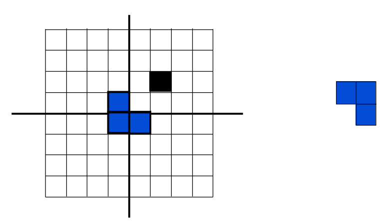
- The 2 x 2 board can be tiled regardless of which one of the four pieces has been omitted

4 x 4 Case



- Divide the 4 x 4 board into four 2 x 2 sub-boards
- One of the four sub-boards has the missing piece
  - By the I.H., that sub-board can be tiled since it is a 2 x 2 board with a missing piece
- Tile the center squares of the three remaining sub-boards as shown
  - This leaves three 2 x 2 boards, each with a missing piece
  - We know these can be tiled by the Induction Hypothesis

2^{k+1} x 2^{k+1} case



- Divide board into four sub-boards and tile the center squares of the three complete sub-boards
- The remaining portions of the sub-boards can be tiled by the I.H. (which assumes we can tile 2^k x 2^k boards)

When Induction Fails

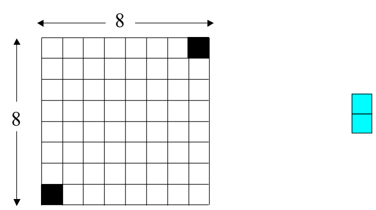
- Sometimes an inductive proof strategy for some proposition may fail
- This does not necessarily mean that the proposition is wrong
  - It may just mean that the particular inductive strategy you are using is the wrong choice
- A different induction hypothesis (or a different proof strategy altogether) may succeed

Tiling Example (Poor Strategy)

Let's try a different induction strategy

- Proposition
  - Any n x n board with one missing square can be tiled
- Problem
  - A 3 x 3 board with one missing square has 8 remaining squares, but our tile has 3 squares; tiling is impossible
- Thus, any attempt to give an inductive proof of this proposition *must fail*
- Note that this failed proof does not tell us anything about the 8x8 case

A Seemingly Similar Tiling Problem



- A chessboard has opposite corners cut out of it. Can the remaining board be tiled using tiles of the shape shown in the picture (rotation allowed)?
- Induction fails here. Why? (Well…for one thing, this board can't be tiled with dominos.)

Strong Induction

- We want to prove that some property P holds for all n
- Weak induction
  - P(0): Show that property P is true for 0
  - P(k) ⇒ P(k+1): Show that if property P is true for k, it is true for k+1
  - Conclude that P(n) holds for all n
- Strong induction
  - P(0): Show that property P is true for 0
  - P(0) and P(1) and … and P(k) ⇒ P(k+1): show that if P is true for numbers less than or equal to k, it is true for k+1
  - Conclude that P(n) holds for all n
- Both proof techniques are equally powerful

Conclusion

- Induction is a powerful proof technique
- Recursion is a powerful programming technique
- Induction and recursion are closely related
  - We can use induction to prove correctness and complexity results about recursive programs