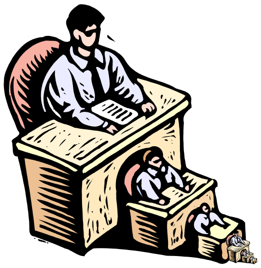


Recursion

Lecture 3
CS211 – Summer 2007

Announcements

- For extra Java help
  - Lots of consulting/office-hours available
  - Can set up individual meetings with TAs via email
- Check soon that you are in CMS
- Academic Integrity Note
  - Assignment #1 must be done <u>individually</u>
  - We treat AI violations seriously
  - The AI hearing process is unpleasant
    - Please help us avoid this process by maintaining Academic Integrity
  - We test all pairs of submitted programming assignments for similarity
    - Similarities are caught even if variables are renamed

Follow-up from yesterday…

- The `String` class is special
  - It's the only class that is allowed to overload operators
    - E.g. `String s = "x" + "y"`
    - No other class is allowed to do this

      Widget ~~w = w1 + w2~~
  - String objects are *immutable*: it is not possible to change the contents of a `String` object after it has been constructed
  - If the same string literal appears multiple times in a program, the compiler *might* create only one object as an *optimization*

`"xy" == "xy"`	`"xy".equals("xy")`
true/false?	true
`"xy" == "x" + "y"`	`"xy".equals("x" + "y")`
true/false?	true
`"xy" == new String("xy")`	`"xy".equals(new String("xy"))`
false	true

Recursion Overview

- Recursion is a powerful technique for specifying functions, sets, and programs
- Example recursively-defined functions and programs
  - factorial
  - combinations
  - exponentiation (raising to an integer power)
- Example recursively-defined sets
  - grammars
  - expressions
  - data structures (lists, trees, ...)

The Factorial Function (n!)

- Define n! = n·(n–1)·(n–2)···3·2·1   read: "n factorial"
  - E.g., 3! = 3·2·1 = 6
- By convention, 0! = 1
- The function int → int that gives n! on input n is called the factorial function
- n! is the number of permutations of n distinct objects
  - There is just one permutation of one object. 1! = 1
  - There are two permutations of two objects: 2! = 2

    1 2   2 1
  - There are six permutations of three objects: 3! = 6

    1 2 3   1 3 2   2 1 3   2 3 1   3 1 2   3 2 1
- If n > 0, n! = n·(n – 1)!

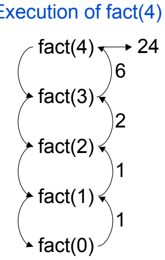
A Recursive Program

0! = 1

n! = n·(n–1)!, n > 0

```
static int fact(int n) {
   if (n == 0) return 1;
   else return n*fact(n-1);
}
```

Execution of fact(4)



General Approach to Writing Recursive Functions

1. Try to find a parameter, say n, such that the solution for n can be obtained by combining solutions to the *same problem using smaller values of n* (e.g., (n-1)!)

2. Find *base case(s)* – small values of n for which you can just write down the solution (e.g., 0! = 1)

3. Verify that, for any valid value of n, applying the reduction of step 1 repeatedly will ultimately hit one of the base cases

The Fibonacci Function

- Mathematical definition:
  - fib(0) = 0
  - fib(1) = 1
  - fib(n) = fib(n – 1) + fib(n – 2), n ≥ 2

  (two base cases!)

- Fibonacci sequence: 0, 1, 1, 2, 3, 5, 8, 13, …

```
static int fib(int n) {
   if (n == 0) return 0;
   else if (n == 1) return 1;
   else return fib(n-1) + fib(n-2);
}
```

Fibonacci (Leonardo Pisano) 1170–1240?

Statue in Pisa, Italy
Giovanni Paganucci
1863

Recursive Execution

```
static int fib(int n) {
   if (n == 0) return 0;
   else if (n == 1) return 1;
   else return fib(n-1) + fib(n-2);
}
```

Execution of fib(4):

- fib(4)
  - fib(3)
    - fib(2)
      - fib(1)
      - fib(0)
    - fib(1)
  - fib(2)
    - fib(1)
    - fib(0)

Recursive vs. iterative solution

```
static int fib(int n) {
   if (n == 0) return 0;
   else if (n == 1) return 1;
   else return fib(n-1) + fib(n-2);
}
```

```
Static int iterative_fib(int n) {
    if (n == 1 || n == 2)
        return 1;
    int last_num = 1, result = 1;
    for (int i = 2; i < n; i++) {
        int temp = result;
        result += last_num;
        last_num = temp;
    }
    return result;
}
```


Combinations (a.k.a. Binomial Coefficients)

- How many ways can you choose r items from a set of n distinct elements? $\binom{n}{r}$ "n choose r"

  $\binom{5}{2}$ = number of 2-element subsets of {A,B,C,D,E}

  2-element subsets containing A: $\binom{4}{1}$
  {A,B}, {A,C}, {A,D}, {A,E}
  2-element subsets not containing A: $\binom{4}{2}$
  {B,C},{B,D},{B,E},{C,D},{C,E},{D,E}

- Therefore, $\binom{5}{2} = \binom{4}{1} + \binom{4}{2}$

Combinations

$$\binom{n}{r} = \binom{n-1}{r} + \binom{n-1}{r-1}, \quad n > r > 0$$
$$\binom{n}{n} = 1$$
$$\binom{n}{0} = 1$$

Can also show that $\binom{n}{r} = \frac{n!}{r!(n-r)!}$

$$\binom{0}{0}$$
$$\binom{1}{0} \quad \binom{1}{1}$$
$$\binom{2}{0} \quad \binom{2}{1} \quad \binom{2}{2}$$
$$\binom{3}{0} \quad \binom{3}{1} \quad \binom{3}{2} \quad \binom{3}{3}$$
$$\binom{4}{0} \quad \binom{4}{1} \quad \binom{4}{2} \quad \binom{4}{3} \quad \binom{4}{4}$$

Pascal's triangle =

1
1 1
1 2 1
1 3 3 1
1 4 6 4 1

Binomial Coefficients

- Combinations are also called *binomial coefficients* because they appear as coefficients in the expansion of the binomial power $(x+y)^n$:

$$(x + y)^n = \binom{n}{0}x^n + \binom{n}{1} x^{n-1}y + \binom{n}{2} x^{n-2}y^2 + \cdots + \binom{n}{n} y^n$$

$$= \sum_{i=0}^{n} \binom{n}{i} x^{n-i}y^i$$

Combinations Have Two Base Cases

$$\binom{n}{r} = \binom{n-1}{r} + \binom{n-1}{r-1}, \quad n > r > 0$$

$$\binom{n}{n} = 1$$

$$\binom{n}{0} = 1$$

Two base cases

- Coming up with right base cases can be tricky!
- General idea:
  - Determine argument values for which recursive case does not apply
  - Introduce a base case for each one of these

Recursive Program for Combinations

$$\binom{n}{r} = \binom{n-1}{r} + \binom{n-1}{r-1}, \quad n > r > 0$$

$$\binom{n}{n} = 1$$

$$\binom{n}{0} = 1$$

```
static int combs(int n, int r) {    //assume n>=r>=0
   if (r == 0 || r == n) return 1; //base cases
   else return combs(n-1,r) + combs(n-1,r-1);
}
```


Positive Integer Powers

- $a^n = a \cdot a \cdot a \cdots a$ (n times)
- Alternate description:
  - $a^0 = 1$
  - $a^{n+1} = a \cdot a^n$

```
static int power(int a, int n)
{
   if (n == 0) return 1;
   else return a*power(a,n-1);
}
```


A Smarter Version

- Power computation:
  - $a^0 = 1$
  - If n is nonzero and even, $a^n = (a^{n/2})^2$
  - If n is odd, $a^n = a \cdot (a^{n/2})^2$
    - Java note: If x and y are integers, "x/y" returns the integer part of the quotient
- Example:
  $a^5 = a \cdot (a^{5/2})^2 = a \cdot (a^2)^2 = a \cdot ((a^{2/2})^2)^2 = a \cdot (a^2)^2$
  Note: this requires 3 multiplications rather than 5!
- What if n were larger?
  - Savings would be more significant
- This is much faster than the straightforward computation
  - Straightforward computation: n multiplications
  - Smarter computation: log(n) multiplications

Smarter Version in Java

- n = 0: $a^0 = 1$
- n nonzero and even: $a^n = (a^{n/2})^2$
- n nonzero and odd: $a^n = a \cdot (a^{n/2})^2$

parameters

local variable

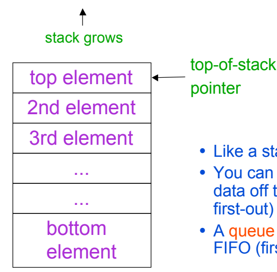
```
static int power(int a, int n) {
   if (n == 0) return 1;
   int halfPower = power(a,n/2);
   if (n%2 == 0) return halfPower*halfPower;
   return halfPower*halfPower*a;
}
```

- The method has two parameters and a local variable
- Why aren't these overwritten on recursive calls?

How the compiler implements recursive methods

- Key idea:
  - Use a stack to remember parameters and local variables across recursive calls
  - Each method invocation gets its own stack frame
- A stack frame contains storage for
  - Local variables of method
  - Parameters of method
  - Return info (return address and return value)
  - Perhaps other bookkeeping info

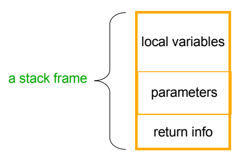
Stacks



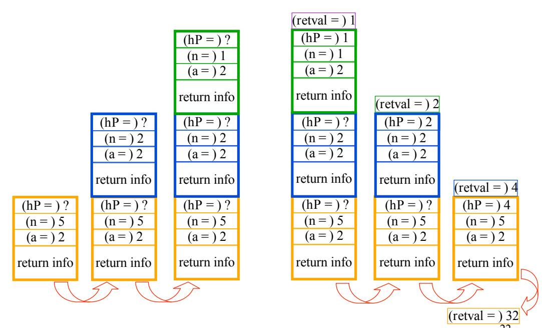
- Like a stack of plates
- You can push data on top or pop data off the top in a LIFO (last-in-first-out) fashion
- A queue is similar, except it is FIFO (first-in-first-out)

Stack Frame

- A new stack frame is pushed with each recursive call
- The stack is popped when the method returns
  - Leaving a return value (if there is one) on top of the stack



Example: power(2, 5)

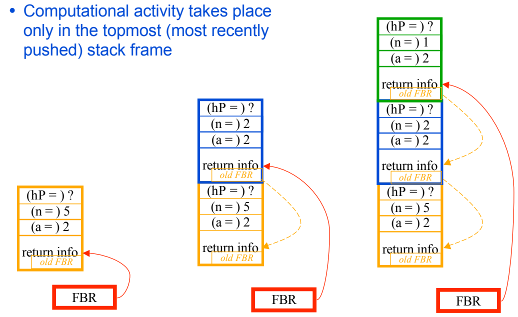


How Do We Keep Track?

- At any point in execution, many invocations of *power* may be in existence
  - Many stack frames (all for *power*) may be in Stack
  - Thus there may be several different versions of the variables *a* and *n*
- How does processor know which location is relevant at a given point in the computation?
  - Frame Base Register
    - When a method is invoked, a frame is created for that method invocation, and FBR is set to point to that frame
    - When the invocation returns, FBR is restored to what it was before the invocation
- How does machine know what value to restore in the FBR?
  - This is part of the return info in the stack frame

FBR

- Computational activity takes place only in the topmost (most recently pushed) stack frame



Iteration or recursion?

- Some languages do not support recursion, others do not support iteration
  - But many modern languages support both
- How to choose?
  - Which is clearer? Which is more intuitive? (often recursion)
  - Which is faster? Which uses less memory? (often iteration)
- Recursion involves some overhead
  - Memory overhead: stack frames
  - Execution time overhead: method calls

Conclusion

- Recursion is a convenient and powerful way to define functions
- Problems that seem insurmountable can often be solved in a “divide-and-conquer” fashion:
  - Reduce a big problem to smaller problems of the same kind, solve the smaller problems
  - Recombine the solutions to smaller problems to form solution for big problem