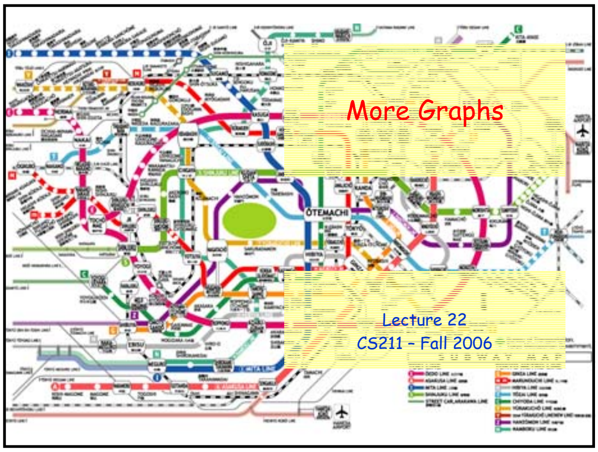


More Graphs

Lecture 22

CS211 – Fall 2006

Adjacency Matrix or Adjacency List?

n = number of vertices

m = number of edges

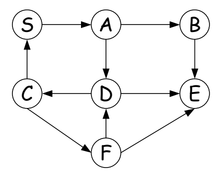
m_u = number of edges leaving u

- Adjacency Matrix
  - Uses space $O(n^2)$
  - Can iterate over all edges in time $O(n^2)$
  - Can answer "Is there an edge from u to v?" in O(1) time
  - Better for dense (i.e., lots of edges) graphs
- Adjacency List
  - Uses space O(m+n)
  - Can iterate over all edges in time O(m+n)
  - Can answer "Is there an edge from u to v?" in $O(m_u)$ time
  - Better for sparse (i.e., fewer edges) graphs

Goal: Find Shortest Path in a Graph

- Finding the shortest (min-cost) path in a graph is a problem that occurs often
  - Find the least-cost route between Ithaca and West Lafayette, IN
  - Result depends on our notion of cost
    - Least mileage
    - Least time
    - Cheapest
    - Least boring
  - All of these "costs" can be represented as edge costs on a graph
- How do we find a shortest path?

Shortest Paths for Unweighted Graphs




```
bfsDistance(s):
    // s is the start vertex
    // dist[v] is length of s-to-v path
    // Initially dist[v] = ∞ for all v
    dist[s] = 0;
    Q.insert(s);

    while (Q nonempty) {
        v = Q.get();
        for (each w adjacent to v) {
            if (dist[w] == ∞) {
                dist[w] = dist[v]+1;
                Q.insert(w);
            }
        }
    }
```

Analysis for bfsDistance

- How many times can a vertex be placed in the queue?
- How much time for the for-loop?
  - Depends on representation
    - Adjacency Matrix: O(n)
    - Adjacency List: $O(m_v)$
- Time:
  - $O(n^2)$ for adj matrix
  - O(m+n) for adj list

```
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```
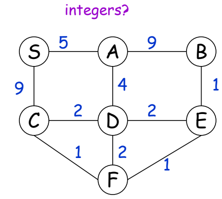

If There are Edge Costs?

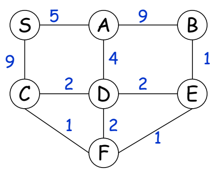
- Idea #1
  - Add false nodes so that all edge costs are 1
  - But what if edge costs are large?
  - What if the costs aren't integers?



- Idea #2
  - Nothing "interesting" happens at the false nodes
    - Can't we just jump ahead to the next real node
  - Intuition
    - Edges are threads; vertices are beads
    - Pick up at s; mark each node as it leave the table
  - Rule: always do the closest-to-s node first
  - Use the array dist[ ] to
    - Report answers
    - *Keep track of what to do next*

Dijkstra's Algorithm

- Intuition
  - Edges are threads; vertices are beads
  - Pick up at s; mark each node as it leave the table
- Note: Negative edge-costs are *not allowed*

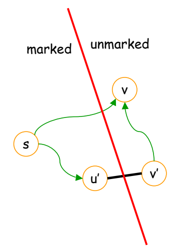


- s is the start vertex
- c(i,j) is the cost from i to j
- Initially, vertices are unmarked
- dist[v] is length of s-to-v path
- Initially, dist[v] = ∞, for all v

```
dijkstra(s):
  dist[s] = 0;
  while (some vertices are unmarked) {
    v = unmarked node with smallest dist;
    Mark v;
    for (each w adj to v) {
      dist[w] = min{dist[w],dist[v]+c(v,w)};
    }
  }
```

Proof for Dijkstra's Algorithm

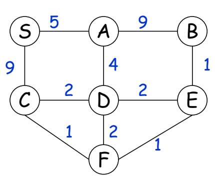
- Claim: When vertex v is marked, dist[v] is the length of the shortest path from s to v



- Proof
  - Suppose there is a shorter path P from s to v
  - Consider the first edge of P that links a marked vertex to an unmarked vertex
    - Such an edge must exist because we know s is marked and v is not
    - Call this edge (u',v')
  - Note that the length of the path from s to u' to v' is less than the length of P
    - Thus v' would be chosen in the algorithm instead of v
    - Contradiction!

Dijkstra's Algorithm using Adj Matrix

- While-loop is done n times
- Within the loop
  - Choosing v takes O(n) time
    - Could do this faster using PQ, but no reason to
  - For-loop takes O(n) time
- Total time = $O(n^2)$



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Dijkstra's Algorithm using Adj List

- Looks like we need a PQ
  - Problem: priorities are updated as algorithm runs
  - Can insert pair (v,dist[v]) in PQ whenever dist[v] is updated
  - At most m things in PQ
- Time O(n + m log m)
- Using a more complicated PQ (e.g., Pairing Heap), time can be brought down to O(m + n log n)

- s is the start vertex
- c(i,j) is the cost from i to j
- Initially, vertices are unmarked
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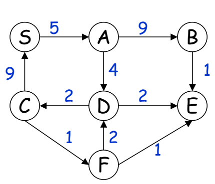
    }
  }
```

Dijkstra's Algorithm for Digraphs

- Algorithm works on both undirected and directed graphs without modification
- As before: Negative edge-costs are *not allowed*



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```

Greedy Algorithms

- Dijkstra's Algorithm is an example of a Greedy Algorithm
- The Greedy Strategy is an algorithm design technique
  - Like Divide & Conquer
- The greedy algorithms are used to solve optimization problems
  - The goal is to find the *best* solution
- Works when the problem has the greedy-choice property
  - A global optimum can be reached by making locally optimum choices

- Problem: Given an amount of money, find the smallest number of coins to make that amount
- Solution: Use a Greedy Algorithm
  - Give as many large coins as you can
- This greedy strategy produces the optimum number of coins for the US coin system
- Different money system ⇒ greedy strategy may fail
  - Example: suppose the US introduces a 4¢ coin