Searching & Asymptotic Complexity

Lecture 12
CS211 – Fall 2006

Announcements

- Prelim 1
  - Occurs at 7:30pm on Thursday (Oct 12) after Fall Break (i.e., 9 days from today)
  - Topics: all material from August & September
    - Includes Interfaces & Comparable
    - Not Searching & Sorting & Asymptotic Complexity (this week's topics)
- Exam conflicts
  - Email Kelly Patwell ASAP
- Prelim 1 review sessions
  - Wed, Oct 11
  - See Exams on course website for more information
  - Individual appointments are available if you cannot attend the review sessions (email *one* TA to arrange appointment)
- Old exams are available for review on the course website
- Sections for Wed, Oct 11, are cancelled
  - This week's sections are the last before Prelim 1

ACSU Announcement

- ACSU 2nd general meeting
  - 5pm Wednesday, Oct 4, in PH 203
  - Talk by John Hopcroft (2005 ACSU Faculty of the Year)
    - Future of CS theory and its applications
  - Free pizza!

What Makes a Good Algorithm?

- Suppose you have two possible algorithms or data structures that basically do the same thing; which is *better*?
- Well… what do we mean by *better*?
  - Faster?
  - Less space?
  - Easier to code?
  - Easier to maintain?
  - Required for homework?
- How do we measure time and space for an algorithm?

Sample Problem: Searching

- Determine if a *sorted* array of integers contains a given integer
- 1st solution: Linear Search (check each element)

```
static boolean find (int[ ] a, int item) {
   for (int i = 0; i < a.length; i++) {
         if (a[i] == item) return true;
         }
   return false;
   }
```

- 2nd solution: Binary Search

```
static boolean find (int[ ] a, int item) {
   int low = 0;
   int high = a.length - 1;
   while (low <=  high) {
         int mid = (low+high)/2;
         if (a[mid] < item)
                 low = mid+1;
         else if (item < a[mid])
                 high = mid - 1;
         else return true;
         }
   return false;
   }
```

Linear Search vs. Binary Search

- Which one is better?
  - Linear Search is easier to program
  - But Binary Search is faster… isn't it?
- How do we measure to show that one is faster than the other
  - Experiment?
  - Proof?
  - But which inputs do we use?
- Simplifying assumption #1: Use the *size* of the input rather than the input itself
  - For our sample search problem, the input size is n+1 where n is the array size
- Simplifying assumption #2: Count the number of "*basic steps*" rather than computing exact times

One Basic Step = One Time Unit

- Basic step:
  - input or output of a scalar value
  - accessing the value of a scalar variable, array element, or field of an object
  - assignment to a variable, array element, or field of an object
  - a single arithmetic or logical operation
  - method invocation (not counting argument evaluation and execution of the method body)
- For a conditional, we count number of basic steps on the branch that is executed
- For a loop, we count number of basic steps in loop body times the number of iterations
- For a method, we count number of basic steps in method body (including steps needed to prepare stack-frame)

Runtime vs. Number of Basic Steps

- But isn't this cheating?
  - The runtime is not the same as the number of basic steps
  - Time per basic step varies depending on computer, on compiler, on details of code…
- Well… yes, it is cheating in a way
  - But the number of basic steps is *proportional* to the actual runtime
- Which is better?
  - n or n^2 time?
  - 100 n or n^2 time?
  - 10,000 n or n^2 time?
- As n gets large, multiplicative constants become less important
- Simplifying assumption #3: Multiplicative constants aren't important

Using Big-O to Hide Constants

- Roughly, f(n) = O(g(n)) means that f(n) grows like g(n) or slower

Definition: O(g(n)) is a *set*, f(n) is a member of this set if and only if there exist constants c and N such that $0 \le f(n) \le c\, g(n)$, for all $n \ge N$

- We *should* write $f(n) \in O(g(n))$
- But by convention, we write f(n) = O(g(n))

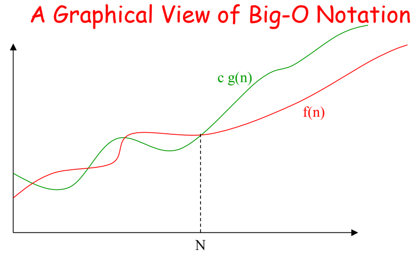
Claim: $n^2 + n = O(n^2)$

We know $n \le n^2$ for $n \ge 1$

So $n^2 + n \le 2\, n^2$ for $n \ge 1$

So by definition,
$n^2 + n = O(n^2)$
for c=2 and N=1

A Graphical View of Big-O Notation



- To prove that f(n) = O(g(n)):
  - Find an N and c such that $0 \le f(n) \le c\, g(n)$, for all $n \ge N$
  - We call the pair (c, N) a *witness pair* for proving that f(n) = O(g(n))

Big-O Examples

Claim: 100 n + log n = O(n)

We know $\log n \le n$ for $n \ge 1$

So $100\, n + \log n \le 101\, n$
for $n \ge 1$

So by definition,
100 n + log n = O(n)
for c=101 and N=1

Claim: $\log_B n = O(\log n)$

Let k = log n

Then $n = 2^k$ and (the subscripts are too messy; switch to board)

Question: Which grows faster: sqrt(n) or log n?

Simple Big-O Examples

- Let $f(n) = 3n^2 + 6n - 7$
  - Claim $f(n) = O(n^2)$
  - Claim $f(n) = O(n^3)$
  - Claim $f(n) = O(n^4)$
  - …
- g(n) = 4n log n + 34 n – 89
  - Claim g(n) = O(n log n)
  - Claim $g(n) = O(n^2)$
- $h(n) = 20 * 2^n + 40$
  - Claim $h(n) = O(2^n)$
- a(n) = 34
  - Claim a(n) = O(1)
- Only the *leading* term (the term that grows most rapidly) matters

Problem-Size Examples

- Suppose we have a computing device that can execute 1000 operations per second; how large a problem can we solve?

	1 second	1 minute	1 hour
n	1000	60,000	3,600,000
n log n	140	4893	200,000
n^2	31	244	1897
$3n^2$	18	144	1096
n^3	10	39	153
2^n	9	15	21

Commonly Seen Time Bounds

O(1)	constant	excellent
O(log n)	logarithmic	excellent
O(n)	linear	good
O(n log n)		pretty good
$O(n^2)$	quadratic	OK
$O(n^3)$	cubic	maybe OK
$O(2^n)$	exponential	too slow

Related Notations

- Big-Omega

Definition:
f(n) is a member of the set $\Omega(g(n))$ if there exists constants c and N such that $0 \le c\, g(n) \le f(n)$, for all $n \ge N$

- Big-Theta

Definition:
f(n) is a member of the set $\Theta(g(n))$ if $f(n) = O(g(n))$ and $f(n) = \Omega(g(n))$

Worst-Case/Expected-Case Bounds

- We can't possibly determine time bounds for all possible inputs of size n
- Simplifying assumption #4: Determine number of steps for either
  - worst-case or
  - expected-case
- Worst-case
  - Determine how much time is needed for the *worst possible* input of size n
- Expected-case
  - Determine how much time is needed *on average* for all inputs of size n

Our Simplifying Assumptions

1. Use the size of the input rather than the input itself
2. Count the number of "basic steps" rather than computing exact times
3. Multiplicative constants aren't important (i.e., use big-O notation)
4. Determine number of steps for either
   1. worst-case or
   2. expected-case

Worst-Case Analysis of Searching

- Linear Search (check each element)

```
static boolean find (int[ ] a, int item) {
   for (int i = 0; i < a.length; i++) {
         if (a[i] == item) return true;
   }
   return false;
}
```

For Linear Search, worst-case time is O(n)

For Binary Search, worst-case time is O(log n)

- Binary Search

```
static boolean find (int[ ] a, int item) {
   int low = 0;
   int high = a.length - 1;
   while (low <=  high) {
         int mid = (low+high)/2;
         if (a[mid] < item)
               low = mid+1;
         else if (item < a[mid])
               high = mid - 1;
         else return true;
         }
   return false;
   }
```

Analysis of Matrix Multiplication

Code for multiplying n-by-n matrices A and B:

```
for (i = 0; i<n; i++)
    for (j = 0; j < n; j++)
        for (k = 0; k < n; k++)
            C[i][j] = C[i][j] + A[i][k] * B[k][j];
```

- By convention, matrix problems are measured in terms of n, the number of rows and columns
  - Note that the input size is really $2n^2$, not n
  - Worst-case time is $O(n^3)$
  - Expected-case time is also $O(n^3)$

Remarks

- Once you get the hang of this, you can quickly zero in on what is relevant for determining asymptotic complexity
  - For example, you can usually ignore everything that is not in the innermost loop. Why?
- Main difficulty:
  - Determining runtime for recursive programs

Why Bother with Runtime Analysis?

- Computers are so fast these days that we can do whatever we want using just simple algorithms and data structures, can't we?
- Well…not really; data-structure/algorithm improvements can be a *very big* win
- Scenario:
  - A runs in n^2 msec
  - A' runs in $n^2/10$ msec
  - B runs in 10 n log n msec

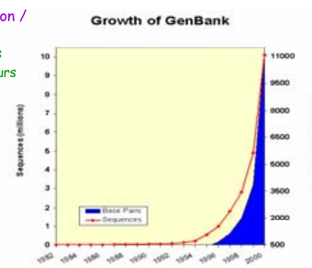
- Problem of size $n=10^3$
  - A: 10^3 sec ≈ 17 minutes
  - A': 10^2 sec ≈ 1.7 minutes
  - B: 10^2 sec ≈ 1.7 minutes
- Problem of size $n=10^6$
  - A: 10^9 sec ≈ 30 years
  - A': 10^8 sec ≈ 3 years
  - B: 2×10^5 sec ≈ 2 days

1 day = 86,400 sec ≈ 10^5 sec

1,000 days ≈ 3 years

Algorithms for the Human Genome

- Human genome
  = 3.5 billion nucleotides
  ~ 1 Gb
  - @ 1 base-pair instruction / μsec
    - $n^2 \Rightarrow$ 388445 years
    - $n \log n \Rightarrow$ 30.824 hours
    - $n \Rightarrow$ 1 hour



Limitations of Runtime Analysis

- Big-O can hide a large constant
  - Example: Selection
  - Example: small problems
- The specific problem you want to solve may not be the worst case
  - Example: Simplex method for linear programming
- Your program may not be run often enough to make analysis worthwhile
  - Example: one-shot vs. every day
- You may be analyzing and improving the wrong part of the program
  - *Very* common situation
  - Should use *profiling* tools

Summary

- Asymptotic complexity
  - Used to measure of time (or space) required by an algorithm
  - Measure of the *algorithm*, not the *problem*
- Searching a sorted array
  - Linear search: O(n) worst-case time
  - Binary search: O(log n) worst-case time
- Matrix operations:
  - Note: n = number-of-rows = number-of-columns
  - Matrix-vector product: $O(n^2)$ worst-case time
  - Matrix-matrix multiplication: $O(n^3)$ worst-case time