Induction

Lecture 5
CS211 – Fall 2006

Announcements

- North Campus consulting is available on Sundays in the RPCC computer lab
  - Check the course website under consulting for times
  - Consultants will have a "tent card"
- We are still checking on West Campus consulting
- Two sections have been split because of crowding
  - Tue 1:25 (section 2)
  - Wed 12:20 (section 4)
  - See the course website
- Reminder
  - A1 is due at 4:30 today
  - Don't wait until 4:29 to discover that you are not on CMS
- A2 (due Wed 9/20) should appear online before the weekend

Overview

- Recursion
  - A programming strategy that solves a problem by reducing it to simpler or smaller instance(s) of the same problem
- Induction
  - A mathematical strategy for proving statements about natural numbers 0,1,2,... (or more generally, about inductively defined objects)
- Induction and recursion are very closely related

Defining Functions

- It is often useful to write a given function in different ways
  - Let $S : int \rightarrow int$ be the function where S(n) is the sum of the integers from 0 to n. E.g.,
    $S(0) = 0$ $\quad\quad$ $S(3) = 0+1+2+3 = 6$
  - Definition: iterative form
    - $S(n) = 0+1+ \ldots + n$
  - Another characterization: closed form
    - $S(n) = n(n+1)/2$

Sum of Squares

- Here is a more complex example.
  - Let $SQ : int \rightarrow int$ be the function that gives the sum of the squares of integers from 0 to n. E.g.,
    $SQ(0) = 0 \quad SQ(3) = 0^2 + 1^2 + 2^2 + 3^2 = 14$
- Definition (iterative form): $SQ(n) = 0^2 + 1^2 + \ldots + n^2$
- Is there an equivalent closed-form expression?

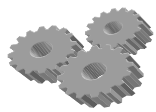
Closed-Form Expression for SQ(n)

- Sum of integers between 0 through n was n(n+1)/2 which is a *quadratic* in n
- Inspired guess: perhaps sum of squares of integers between 0 through n is a *cubic* in n

- Conjecture: $SQ(n) = an^3+bn^2+cn+d$
  where a, b, c, d are unknown coefficients
- How can we find the values of the four unknowns?
  - Idea: Use any 4 values of n to generate 4 linear equations, and then solve

Finding Coefficients

$$SQ(n) = 0^2+1^2+...+n^2 = an^3+bn^2+cn+d$$



- Use $n = 0, 1, 2, 3$

$SQ(0) = 0 = a \cdot 0 + b \cdot 0 + c \cdot 0 + d$
$SQ(1) = 1 = a \cdot 1 + b \cdot 1 + c \cdot 1 + d$
$SQ(2) = 5 = a \cdot 8 + b \cdot 4 + c \cdot 2 + d$
$SQ(3) = 14 = a \cdot 27 + b \cdot 9 + c \cdot 3 + d$

Solve these 4 equations to get
$a = 1/3$, $b = 1/2$, $c = 1/6$, $d = 0$

Is the Formula Correct?

- This suggests

$$SQ(n) = 0^2 + 1^2 + ... + n^2 = n^3/3 + n^2/2 + n/6 = n(n+1)(2n+1)/6$$

- Question: How do we know this closed-form solution is true for all values of n?
  - Remember, we only used n = 0,1,2,3 to determine these coefficients
  - We do not know that the closed-form expression is valid for other values of n

One Approach

- Try a few other values of n to see if they work.
  - Try n = 5: $SQ(n) = 0+1+4+9+16+25 = 55$
  - Closed-form expression: $5 \cdot 6 \cdot 11/6 = 55$
  - Works!
- Try some more values…
- Problem: We can never prove validity of the closed-form solution for all values of n this way since there are an infinite number of values of n

A Recursive Definition

- To solve this problem, let's express SQ(n) in a different way:
  $SQ(n) = \boxed{0^2 + 1^2 + ... + (n-1)^2} + n^2$
  - The part in the box is just SQ(n-1)
- This leads to the following recursive definition
  $SQ(0) = 0$ ← Base Case
  $SQ(n) = SQ(n-1) + n^2, \ n > 0$ ← Recursive Case
- Thus,
  $SQ(4) = SQ(3) + 4^2 = SQ(2) + 3^2 + 4^2 = SQ(1) + 2^2 + 3^2 + 4^2$
  $= SQ(0) + 1^2 + 2^2 + 3^2 + 4^2 = 0 + 1^2 + 2^2 + 3^2 + 4^2$

Are These Two Functions Equal?

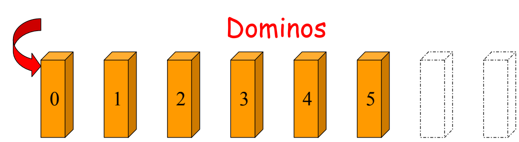
- SQ_r (r = recursive)

$$SQ_r(0) = 0$$
$$SQ_r(n) = SQ_r(n-1) + n^2, \quad n > 0$$

- SQ_c (c = closed-form)

$$SQ_c(n) = n(n+1)(2n+1)/6$$

Dominos



- Assume equally spaced dominos, and assume that spacing between dominos is less than domino length
- How would you argue that all dominos would fall?
- Dumb argument:
  - Domino 0 falls because we push it over
  - Domino 0 hits domino 1, therefore domino 1 falls
  - Domino 1 hits domino 2, therefore domino 2 falls
  - Domino 2 hits domino 3, therefore domino 3 falls
  - …
- Is there a more compact argument we can make?

Better Argument

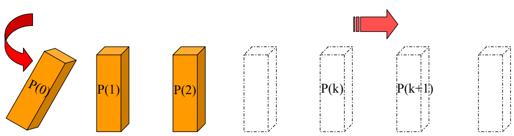
- Argument:
  - Domino 0 falls because we push it over (Base Case or Basis)
  - Assume that domino k falls over (Induction Hypothesis)
  - Because domino k's length is larger than inter-domino spacing, it will knock over domino k+1 (Inductive Step)
  - Because we could have picked any domino to be the k^{th} one, we conclude that all dominoes will fall over (Conclusion)
- This is an inductive argument
- This version is called *weak induction*
  - There is also *strong induction* (later)
- Not only is this argument more compact, it works for an arbitrary number of dominoes!

Weak Induction over Integers

- We want to prove that some property P(n) holds for all integers $n \geq 0$
- Inductive argument
  - Basis: Show that property P is true for 0
  - Induction Hypothesis: Assume that P(k) is true for an *unspecified* integer k
  - Inductive Step: Under this assumption, show that P(k+1) is true
  - Conclusion: Because we could have picked any k, we conclude that P(n) holds for all integers $n \geq 0$

$SQ_r(n) = SQ_c(n)$ for all n?

- Define P(n) as $SQ_r(n) = SQ_c(n)$



- Prove P(0)
- Assume P(k) for unspecified k, and then prove P(k+1) under this assumption

Proof (by Induction)

Recall: $SQ_r(0) = 0$
$SQ_r(n) = SQ_r(n-1) + n^2, \quad n > 0$

$SQ_c(n) = n(n+1)(2n+1)/6$

Let P(n) be the proposition that $SQ_r(n) = SQ_c(n)$

- Basis: P(0) holds because $SQ_r(0) = 0$ and $SQ_c(0) = 0$ by definition
- Induction Hypothesis: Assume $SQ_r(k) = SQ_c(k)$ for some k
- Inductive Step:

$SQ_r(k+1)$	$= SQ_r(k) + (k+1)^2$	by definition
	$= SQ_c(k) + (k+1)^2$	by I.H.
	$= k(k+1)(2k+1)/6 + (k+1)^2$	by definition
	$= (k+1)(k+2)(2k+3)/6$	algebra
	$= SQ_c(k+1)$	by definition

- Conclusion: $SQ_r(n) = SQ_c(n)$ for all $n \geq 0$

Another Example

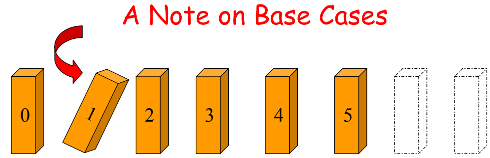
Prove that $0+1+\ldots+n = n(n+1)/2$

- Basis: Obviously holds for $n = 0$
- Induction Hypothesis: Assume $0+1+\ldots+k = k(k+1)/2$
- Inductive Step:

$0+1+\ldots+(k+1)$	$= [0+1+\ldots+k] + (k+1)$	by def
	$= k(k+1)/2 + (k+1)$	I.H.
	$= (k+1)(k+2)/2$	algebra

- Conclusion: $0+1+\ldots+n = n(n+1)/2$ for all $n \geq 0$

A Note on Base Cases



- Sometimes we are interested in showing some proposition is true for integers $\geq b$
- Intuition: we knock over domino b, and dominoes in front get knocked over; not interested in $0,1,\ldots,(b-1)$
- In general, the base case in induction does not have to be 0
- If base case is some integer b
  - Induction proves the proposition for $n = b, b+1, b+2, \ldots$
  - Does not say anything about $n = 0,1,\ldots,b-1$

Weak Induction: Nonzero Base Case

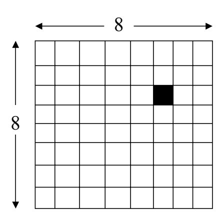
Claim: You can make any amount of postage above 8¢ with some combination of 3¢ and 5¢ stamps

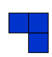
- Basis: True for 8¢: 8 = 3 + 5
- Induction Hypothesis: Suppose true for some $k \geq 8$
- Inductive Step:
  - If used a 5¢ stamp to make k, replace it by two 3¢ stamps. Get k+1.
  - If did not use a 5¢ stamp to make k, must have used at least three 3¢ stamps. Replace three 3¢ stamps by two 5¢ stamps. Get k+1.
- Conclusion: Any amount of postage above 8¢ can be made with some combination of 3¢ and 5¢ stamps

What are the "Dominos"?

- In some problems, it can be tricky to determine how to set up the induction
- This is particularly true for geometric problems that can be attacked using induction

A Tiling Problem





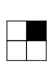
- A chessboard has one square cut out of it
- Can the remaining board be tiled using tiles of the shape shown in the picture (rotation allowed)?
- Not obvious that we can use induction!

Proof Outline

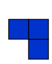
Consider boards of size $2^n \times 2^n$ for n = 1,2,…

- Basis: Show that tiling is possible for 2 x 2 board
- Induction Hypothesis: Assume the $2^k \times 2^k$ board can be tiled
- Inductive Step: Using I.H. show that the $2^{k+1} \times 2^{k+1}$ board can be tiled
- Conclusion: Any $2^n \times 2^n$ board can be tiled, n = 1,2,…
  - Our chessboard (8 x 8) is a special case of this argument
  - We have proved the 8 x 8 special case by solving a more general problem!

Basis

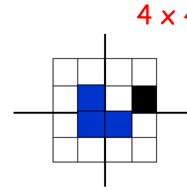


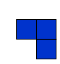
2 x 2 board



- The 2 x 2 board can be tiled regardless of which one of the four pieces has been omitted

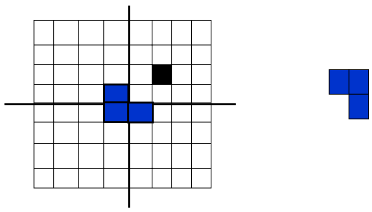
4 x 4 Case





- Divide the 4 x 4 board into four 2 x 2 sub-boards
- One of the four sub-boards has the missing piece
  - By the I.H., that sub-board can be tiled since it is a 2 x 2 board with a missing piece
- Tile the center squares of the three remaining sub-boards as shown
  - This leaves three 2 x 2 boards, each with a missing piece
  - We know these can be tiled by the Induction Hypothesis

2^{k+1} x 2^{k+1} case



- Divide board into four sub-boards and tile the center squares of the three complete sub-boards
- The remaining portions of the sub-boards can be tiled by the I.H. (which assumes we can tile 2^k x 2^k boards)

When Induction Fails

- Sometimes an inductive proof strategy for some proposition may fail
- This does not necessarily mean that the proposition is wrong
  - It may just mean that the particular inductive strategy you are using is the wrong choice
- A different induction hypothesis (or a different proof strategy altogether) may succeed

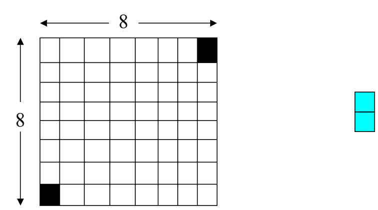
Tiling Example (Poor Strategy)

Let's try a different induction strategy

- Proposition
  - Any n x n board with one missing square can be tiled
- Problem
  - A 3 x 3 board with one missing square has 8 remaining squares, but our tile has 3 squares; tiling is impossible
- Thus, any attempt to give an inductive proof of this proposition *must fail*

- Note that this failed proof does not tell us anything about the 8x8 case

A Seemingly Similar Tiling Problem



- A chessboard has opposite corners cut out of it. Can the remaining board be tiled using tiles of the shape shown in the picture (rotation allowed)?
- Induction fails here. Why? (Well… for one thing, this board can't be tiled with dominos.)

Strong Induction

- We want to prove that some property P holds for all n
- Weak induction
  - P(0): Show that property P is true for 0
  - P(k) $\Rightarrow$ P(k+1): Show that if property P is true for k, it is true for k+1
  - Conclude that P(n) holds for all n
- Strong induction
  - P(0): Show that property P is true for 0
  - P(0) and P(1) and … and P(k) $\Rightarrow$ P(k+1): show that if P is true for numbers less than or equal to k, it is true for k+1
  - Conclude that P(n) holds for all n
- Both proof techniques are equally powerful

Conclusion

- Induction is a powerful proof technique
- Recursion is a powerful programming technique
- Induction and recursion are closely related
  - We can use induction to prove correctness and complexity results about recursive programs