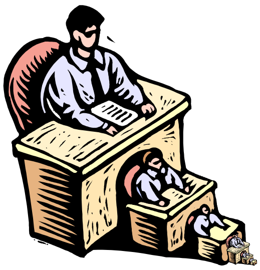


Recursion

Lecture 3
CS211 – Fall 2006

Announcements

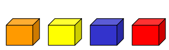
- For extra Java help
  - Lots of consulting/office-hours are available
    - General Java-help is more easily available in week *before* assignment is due
  - Can set up individual meetings with TAs via email
- Check (soon!) that you appear correctly within CMS
  - Report problems to Kelly Patwell (Course Administrator)
- ACSU Event: Helen Newman lanes; 7-9pm tonight (?)
  - Free bowling; free pizza!
- Would you make use of consulting if it were
  - on North Campus (RPU)?
  - on West Campus?
- Academic Integrity Note
  - We treat AI violations seriously
  - The AI Hearing process is unpleasant
    - Please help us avoid this process by maintaining Academic Integrity
  - We test all pairs of submitted programming assignments for similarity
    - Similarities are caught even if variables are renamed

Recursion Overview

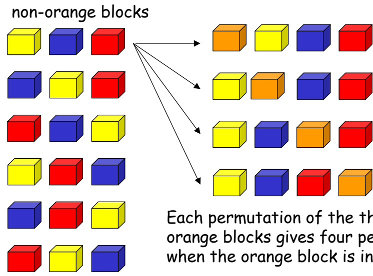
- Recursion is a powerful technique for specifying functions, sets, and programs
- Example recursively-defined functions and programs
  - factorial
  - combinations
  - exponentiation (raising to an integer power)
- Example recursively-defined sets
  - grammars
  - expressions
  - data structures (lists, trees, ...)

The Factorial Function (n!)

- Define $n! = n\cdot(n-1)\cdot(n-2)\cdots 3\cdot 2\cdot 1$ read: "n factorial"
  - E.g., $3! = 3\cdot 2\cdot 1 = 6$
- By convention, $0! = 1$
- The function int → int that gives n! on input n is called the factorial function
- n! is the number of permutations of n distinct objects
  - There is just one permutation of one object. $1! = 1$
  - There are two permutations of two objects: $2! = 2$
    1 2   2 1
  - There are six permutations of three objects: $3! = 6$
    1 2 3   1 3 2   2 1 3   2 3 1   3 1 2   3 2 1
- If $n > 0$, $n! = n\cdot(n-1)!$

Permutations of 

Permutations of non-orange blocks



Each permutation of the three non-orange blocks gives four permutations when the orange block is included

Total number = 4·6 = 24 = 4!

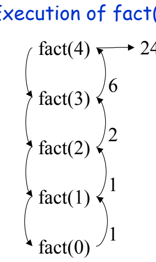
A Recursive Program

$0! = 1$

$n! = n\cdot(n-1)!,\ n > 0$

```
static int fact(int n) {
   if (n == 0) return 1;
   else return n*fact(n-1);
}
```

Execution of fact(4)



General Approach to Writing Recursive Functions

1. Try to find a parameter, say n, such that the solution for n can be obtained by combining solutions to the *same problem using smaller values of n* (e.g., (n-1)!)
2. Figure out the base case(s) — small values of n for which you can just write down the solution (e.g., 0! = 1)
3. Verify that, for any valid value of n, applying the reduction of step 1 repeatedly will ultimately hit one of the base cases

The Fibonacci Function

- Mathematical definition:
  fib(0) = 0
  fib(1) = 1 ← two base cases!
  fib(n) = fib(n – 1) + fib(n – 2), $n \geq 2$
- Fibonacci sequence: 0, 1, 1, 2, 3, 5, 8, 13, …

```
static int fib(int n) {
   if (n == 0) return 0;
   else if (n == 1) return 1;
   else return fib(n-1) + fib(n-2);
}
```

Fibonacci (Leonardo Pisano) 1170–1240?

Statue in Pisa, Italy
Giovanni Paganucci
1863

Recursive Execution

```
static int fib(int n) {
   if (n == 0) return 0;
   else if (n == 1) return 1;
   else return fib(n-1) + fib(n-2);
}
```

Execution of fib(4):

```
                fib(4)
           /            \
       fib(3)          fib(2)
       /    \          /    \
   fib(2)  fib(1)  fib(1)  fib(0)
   /    \
fib(1) fib(0)
```

Combinations (a.k.a. Binomial Coefficients)

- How many ways can you choose r items from a set S of n distinct elements? $\binom{n}{r}$ "n choose r"

$\binom{5}{2}$ = number of 2-element subsets of S = {A,B,C,D,E}

- 2-element subsets containing A: $\binom{4}{1}$
  {A,B}, {A,C}, {A,D}, {A,E}
- 2-element subsets not containing A: $\binom{4}{2}$
  {B,C},{B,D},{B,E},{C,D},{C,E},{D,E}
- Therefore, $\binom{5}{2} = \binom{4}{1} + \binom{4}{2}$

Combinations

$$\binom{n}{r} = \binom{n-1}{r} + \binom{n-1}{r-1}, \quad n > r > 0$$
$$\binom{n}{n} = 1$$
$$\binom{n}{0} = 1$$

Can also show that $\binom{n}{r} = \dfrac{n!}{r!(n-r)!}$

$$\binom{0}{0}$$
$$\binom{1}{0}\ \binom{1}{1}$$
$$\binom{2}{0}\ \binom{2}{1}\ \binom{2}{2}$$
$$\binom{3}{0}\ \binom{3}{1}\ \binom{3}{2}\ \binom{3}{3}$$
$$\binom{4}{0}\ \binom{4}{1}\ \binom{4}{2}\ \binom{4}{3}\ \binom{4}{4}$$

Pascal's triangle =

```
          1
        1   1
      1   2   1
    1   3   3   1
  1   4   6   4   1
1   5  10  10   5   1
```

Binomial Coefficients

- Combinations are also called *binomial coefficients* because they appear as coefficients in the expansion of the binomial power $(x+y)^n$:

$$(x+y)^n = \binom{n}{0} x^n + \binom{n}{1} x^{n-1}y + \binom{n}{2} x^{n-2}y^2 + \cdots + \binom{n}{n} y^n$$
$$= \sum_{i=0}^{n} \binom{n}{i} x^{n-i}y^i$$

Combinations Have Two Base Cases

$$\binom{n}{r} = \binom{n-1}{r} + \binom{n-1}{r-1}, \quad n > r > 0$$
$$\binom{n}{n} = 1$$
$$\binom{n}{0} = 1$$

Two base cases

- Coming up with right base cases can be tricky!
- General idea:
  - Determine argument values for which recursive case does not apply
  - Introduce a base case for each one of these
- Rule of thumb (not always valid): If you have r recursive calls on right hand side, you may need r base cases.

Recursive Program for Combinations

$$\binom{n}{r} = \binom{n-1}{r} + \binom{n-1}{r-1}, \quad n > r > 0$$
$$\binom{n}{n} = 1$$
$$\binom{n}{0} = 1$$

```
static int combs(int n, int r){    //assume n>=r>=0
   if (r == 0 || r == n) return 1; //base cases
   else return combs(n-1,r) + combs(n-1,r-1);
}
```

Positive Integer Powers

- $a^n = a \cdot a \cdot a \cdots a$ (n times)
- Alternate description:
  - $a^0 = 1$
  - $a^{n+1} = a \cdot a^n$

```
static int power(int a, int n) {
   if (n == 0) return 1;
   else return a*power(a,n-1);
}
```

A Smarter Version

- Power computation:
  - $a^0 = 1$
  - If n is nonzero and even, $a^n = (a^{n/2})^2$
  - If n is odd, $a^n = a \cdot (a^{n/2})^2$
    - Java note: If x and y are integers, "x/y" returns the integer part of the quotient
- Example:
  $a^5 = a \cdot (a^{5/2})^2 = a \cdot (a^2)^2 = a \cdot ((a^{2/2})^2)^2 = a \cdot (a^2)^2$
  Note: this requires 3 multiplications rather than 5!
- What if n were larger?
  - Savings would be more significant
- This is much faster than the straightforward computation
  - Straightforward computation: n multiplications
  - Smarter computation: log(n) multiplications

Smarter Version in Java

- n = 0: $a^0 = 1$
- n nonzero and even: $a^n = (a^{n/2})^2$
- n nonzero and odd: $a^n = a \cdot (a^{n/2})^2$

local variable

parameters

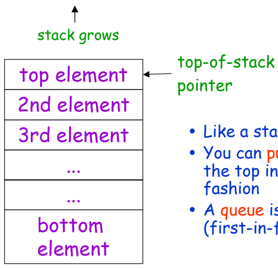
```
static int power(int a, int n) {
   if (n == 0) return 1;
   int halfPower = power(a,n/2);
   if (n%2 == 0) return halfPower*halfPower;
   return halfPower*halfPower*a;
}
```

- The method has two parameters and a local variable
- Why aren't these overwritten on recursive calls?

Implementation of Recursive Methods

- Key idea:
  - Use a stack to remember parameters and local variables across recursive calls
  - Each method invocation gets its own stack frame
- A stack frame contains storage for
  - Local variables of method
  - Parameters of method
  - Return info (return address and return value)
  - Perhaps other bookkeeping info

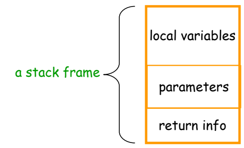
Stacks



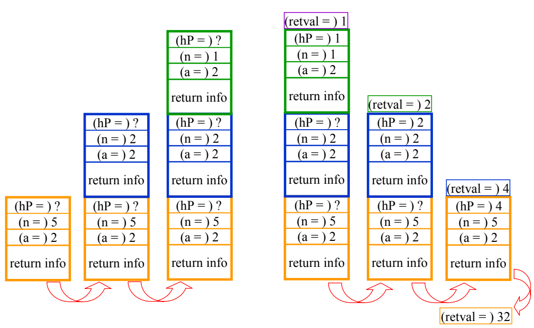
- Like a stack of plates
- You can push data on top or pop data off the top in a LIFO (last-in-first-out) fashion
- A queue is similar, except it is FIFO (first-in-first-out)

Stack Frame

- A new stack frame is pushed with each recursive call
- The stack frame is popped when the method returns
  - Leaving a return value (if there is one) on top of the Stack



Example: power(2, 5)

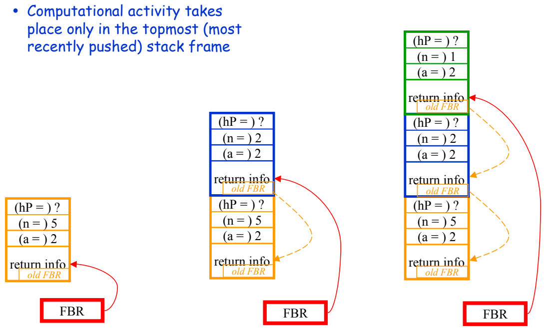


How Do We Keep Track?

- At any point in execution, many invocations of *power* may be in existence
  - Many stack frames (all for *power*) may be in Stack
  - Thus there may be several different versions of the variables *a* and *n*
- How does processor know which location is relevant at a given point in the computation?
- Answer:
  Frame Base Register
  - When a method is invoked, a frame is created for that method invocation, and FBR is set to point to that frame
  - When the invocation returns, FBR is restored to what it was before the invocation
- How does machine know what value to restore in the FBR?
  - This is part of the return info in the stack frame

FBR

- Computational activity takes place only in the topmost (most recently pushed) stack frame



Conclusion

- Recursion is a convenient and powerful way to define functions
- Problems that seem insurmountable can often be solved in a "divide-and-conquer" fashion:
  - Reduce a big problem to smaller problems of the same kind, solve the smaller problems
  - Recombine the solutions to smaller problems to form solution for big problem
- Important application (next lecture): parsing