CS 211

Computers and Programming

http://www.cs.cornell.edu/courses/cs211/2005su

Lecture 3: Recursion

Summer 2005

Announcements

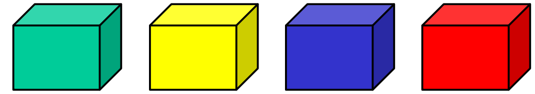
- Small correction to written assignment in problem 1: the summation is over F_i not F_n
- Continue reading Chapter 7
- Quiz on Friday: probably 1 Induction problem and two shortish concept problems.
- If you need a programming partner still, see me after class

Recursion

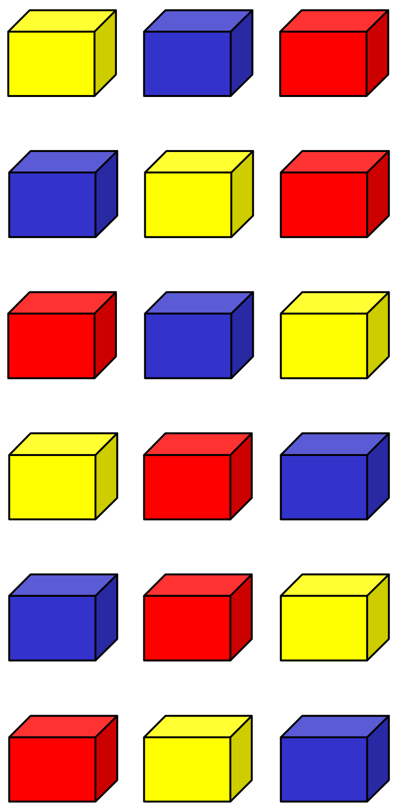
- Recursion is a powerful technique for specifying functions, sets, and programs
- Recursively-defined functions and programs
  - factorial
  - combinations
  - differentiation of polynomials
- Recursively-defined sets
  - grammars
  - expressions
  - data structures (lists, trees, ...)

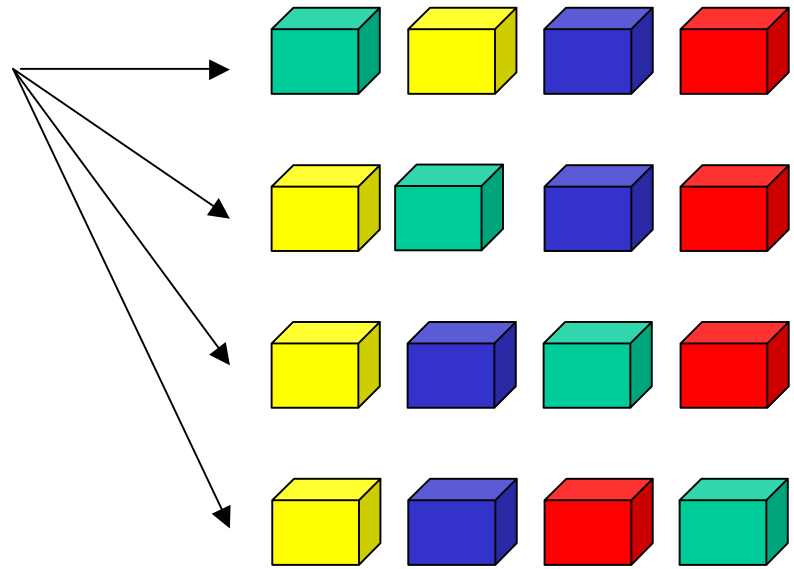
The Factorial Function (n!)

- Define $n! = n\cdot(n-1)\cdot(n-2)\cdots3\cdot2\cdot1$ read: “n factorial”
- E.g., $3! = 3\cdot2\cdot1 = 6$
- By convention, $0! = 1$
- The function int $\rightarrow$ int that gives n! on input n is called the factorial function.
- n! is the number of permutations of n distinct objects
  - There is just one permutation of one object. $1! = 1$
  - There are two permutations of two objects: $2! = 2$
    - 1 2 2 1
  - There are six permutations of three objects: $3! = 6$
    - 1 2 3 1 3 2 2 1 3 2 3 1 3 1 2 3 2 1
- If $n > 1$, $n! = n\cdot(n-1)!$

Permutations of 

Permutations of
non-green blocks





Each permutation of the three non-green blocks gives four permutations of the four blocks.

Total number = $4 \cdot 6 = 24 = 4!$

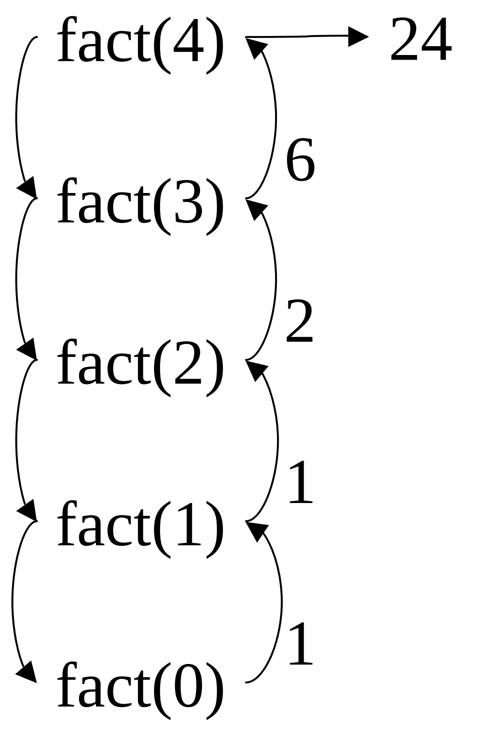
A Recursive Program

$0! = 1$

$n! = n \cdot (n-1)!, \quad n > 0$

```
static int fact(int n) {
    if (n == 0) return 1;
    else return n*fact(n-1);
}
```

Execution of fact(4)



General Approach to Writing Recursive Functions

1. Try to find a parameter, say *n*, such that the solution for *n* can be obtained by combining solutions to the *same* problem with smaller values of *n* (e.g., chess-board tiling, factorial)
2. Figure out the base case(s) -- small values of *n* for which you can just write down the solution (e.g., 0! = 1)
3. Verify that for any value of *n* of interest, applying the reduction of step 1 repeatedly will ultimately hit one of the base cases

The Fibonacci Function

- Mathematical definition:

  $fib(0) = 1$

  $fib(1) = 1$

  two base cases!

  $fib(n) = fib(n-1) + fib(n-2), \quad n = 2$
- Fibonacci sequence: 1, 1, 2, 3, 5, 8, 13, …

```
static int fib(int n) {
   if (n == 0) return 1;
   else if (n == 1) return 1;
   else return fib(n-1) + fib(n-2);
}
```

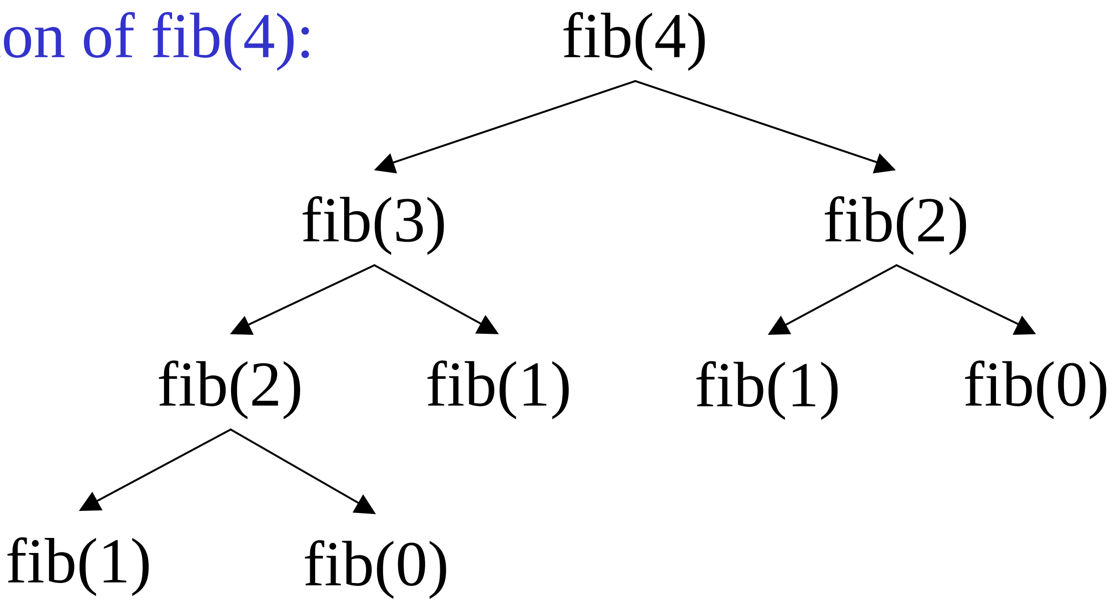
Fibonacci
(Leonardo Pisano,
1170–1240?)

Statue in Pisa, Italy
Giovanni Paganucci,
1863

Recursive Execution

```
static int fib(int n) {
   if (n == 0) return 1;
   else if (n == 1) return 1;
   else return fib(n-1) + fib(n-2);
}
```

Execution of fib(4):



Combinations (a.k.a. Binomial Coefficients)

How many ways can you choose r items from

a set S of n distinct elements? $\binom{n}{r}$ "n choose r"

$\binom{5}{2}$ = number of 2-element subsets of S = {A,B,C,D,E}

- subsets containing A: {A,B}, {A,C}, {A,D},{A,E} $\binom{4}{1}$
- subsets not containing A: {B,C},{B,D},{B,E},{C,D},{C,E},{D,E} $\binom{4}{2}$

Therefore, $\binom{5}{2} = \binom{4}{1} + \binom{4}{2}$

Combinations

$$\binom{n}{r} = \binom{n-1}{r} + \binom{n-1}{r-1}, \quad n > r > 0$$

$$\binom{n}{n} = 1$$

$$\binom{n}{0} = 1$$

- You can also show that $\binom{n}{r} = \dfrac{n!}{r!(n-r)!}$

Combinations

$$\binom{n}{r} = \binom{n-1}{r} + \binom{n-1}{r-1}, \quad n > r > 0$$

$$\binom{n}{n} = 1$$

$$\binom{n}{0} = 1$$

$$
\begin{array}{ccccccccc}
 & & & & \binom{0}{0} & & & & \\
 & & & \binom{1}{0} & & \binom{1}{1} & & & \\
 & & \binom{2}{0} & & \binom{2}{1} & & \binom{2}{2} & & \\
 & \binom{3}{0} & & \binom{3}{1} & & \binom{3}{2} & & \binom{3}{3} & \\
\binom{4}{0} & & \binom{4}{1} & & \binom{4}{2} & & \binom{4}{3} & & \binom{4}{4}
\end{array}
$$

Pascal's triangle

=

$$
\begin{array}{ccccccccccc}
 & & & & & 1 & & & & & \\
 & & & & 1 & & 1 & & & & \\
 & & & 1 & & 2 & & 1 & & & \\
 & & 1 & & 3 & & 3 & & 1 & & \\
 & 1 & & 4 & & 6 & & 4 & & 1 & \\
1 & & 5 & & 10 & & 10 & & 5 & & 1
\end{array}
$$

Combinations

These are also called binomial coefficients because they appear as coefficients in the expansion of the binomial power $(x + y)^n$:

$$(x + y)^n = \binom{n}{0} x^n + \binom{n}{1} x^{n-1}y + \binom{n}{2} x^{n-2}y^2 + \cdots + \binom{n}{n} y^n$$

$$= \sum_{i=0}^{n} \binom{n}{i} x^{n-i}y^i$$

Combinations have two base cases

$$\binom{n}{r} = \binom{n-1}{r} + \binom{n-1}{r-1}, \quad n > r > 0$$

$$\binom{n}{n} = 1$$

$$\binom{n}{0} = 1$$

Two base cases

- Coming up with right base cases can be tricky!
- General idea:
  - Determine argument values for which recursive case does not apply
  - Introduce a base case for each one of these
- Rule of thumb: (not always valid) if you have r recursive calls on right hand side, you may need r base cases.

Recursive Program for Combinations

$$\binom{n}{r} = \binom{n-1}{r} + \binom{n-1}{r-1}, \quad n > r > 0$$

$$\binom{n}{n} = 1$$

$$\binom{n}{0} = 1$$

```
static int combs(int n, int r){     //assume n>=r>=0
   if (r == 0 || r == n) return 1; //base cases
   else return combs(n-1,r) + combs(n-1,r-1);
}
```

Polynomial Differentiation

Inductive cases:

$d(uv)/dx = u\, dv/dx + v\, du/dx$

$d(u+v)/dx = du/dx + dv/dx$

Base cases:

$dx/dx = 1$

$dc/dx = 0$

Example:

$$d(3x)/dx = 3dx/dx + x\, d(3)/dx = 3 \cdot 1 + x \cdot 0 = 3$$

Positive Integer Powers

$a^n = a \cdot a \cdot a \cdots a$ (n times)

Alternative description:

$a^0 = 1$

$a^{n+1} = a \cdot a^n$

```
static int power(int a, int n) {
   if (n == 0) return 1;
   else return a*power(a,n-1);

}
```

A Smarter Version

- Power computation:
  - $a^0 = 1$
  - If n is nonzero and even, $a^n = (a^{n/2})^2$
  - If n is odd, $a^n = a \cdot (a^{n/2})^2$
- Java note: If x and y are integers, “x/y” returns the integer part of the quotient
- Example:

  $a^5 = a \cdot (a^{5/2})^2 = a \cdot (a^2)^2 = a \cdot ((a^{2/2})^2)^2 = a \cdot (a^2)^2$

  Note: this requires 3 multiplications rather than 5!
- What if n were higher?
  - savings would be higher
- This is much faster than the straightforward computation
  - Straightforward computation: n multiplications
  - Smarter computation: log(n) multiplications

Smarter Version in Java

- $n = 0$: $a^0 = 1$
- n nonzero and even: $a^n = (a^{n/2})^2$
- n odd: $a^n = a \cdot (a^{n/2})^2$

```
static int power(int a, int n) {
   if (n == 0) return 1;
   int halfPower = power(a,n/2);
   if (n%2 == 0) return halfPower*halfPower;
   return halfPower*halfPower*a;
}
```

Implementation of Recursive Methods

local variable

parameters

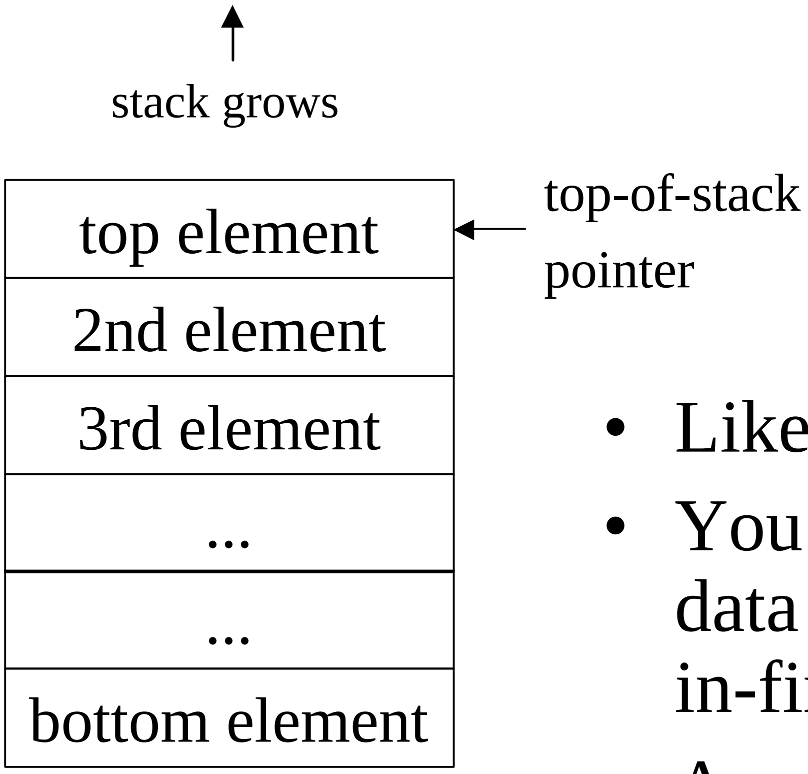
```
static int power(int a, int n) {
   if (n == 0) return 1;
   int halfPower = power(a,n/2);
   if (n%2 == 0) return halfPower*halfPower;
   return halfPower*halfPower*a;
}
```

- The method has two parameters and a local variable
- Why aren’t these overwritten on recursive calls?

Implementation of Recursive Methods

- Key idea:
  - use a stack to remember parameters and local variables across recursive calls
  - each method invocation gets its own stack frame
- A stack frame contains storage for
  - parameters of method
  - local variables of method
  - return address
  - perhaps other bookkeeping info

Stacks

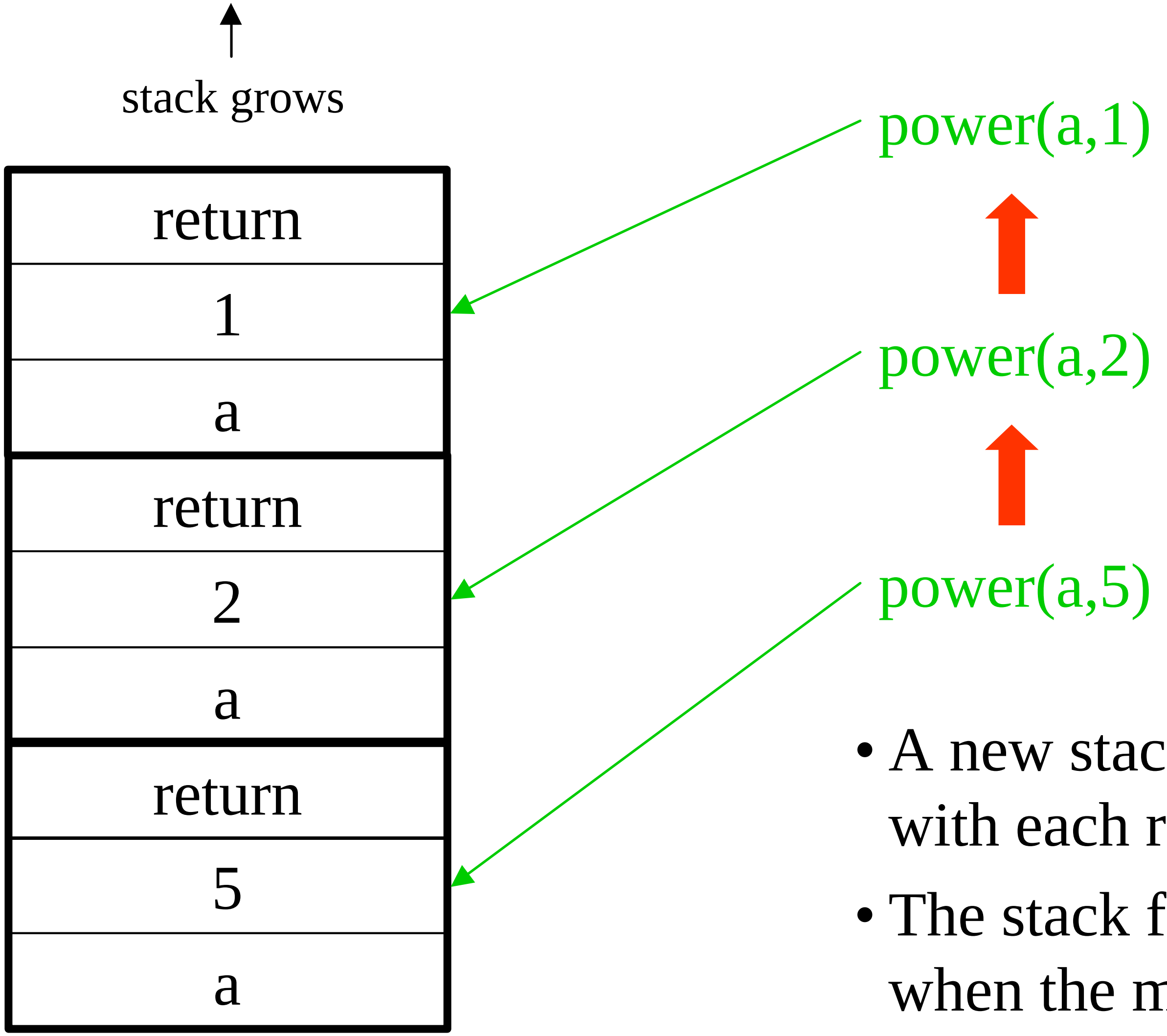


- Like a stack of plates
- You can push data on top or pop data off the top in a LIFO (last-in-first-out) fashion
- A queue is similar, except it is FIFO (first-in-first-out)

java.lang.Stack

• Stack()	Creates an empty Stack
• boolean empty()	Tests if the stack is empty
• E peek()	Looks at the object at the top of the stack without removing it from the stack
• E pop()	Removes the object at the top of the stack and returns that object as the value of the function
• push(E item)	Pushes an item onto the top of the stack
• int search(E o)	Returns the position of the given item on the stack

Stack Frames



- A new stack frame is pushed with each recursive call
- The stack frame is popped when the method returns

Conclusion

- Recursion is a convenient and powerful way to define functions
- Problems that seem insurmountable can often be solved in a "divide-and-conquer" fashion:
  - Reduce a big problem to smaller problems of the same kind, solve the smaller problems
  - Recombine the solutions to smaller problems to form solution for big problem
- Important application (next lecture): parsing of languages